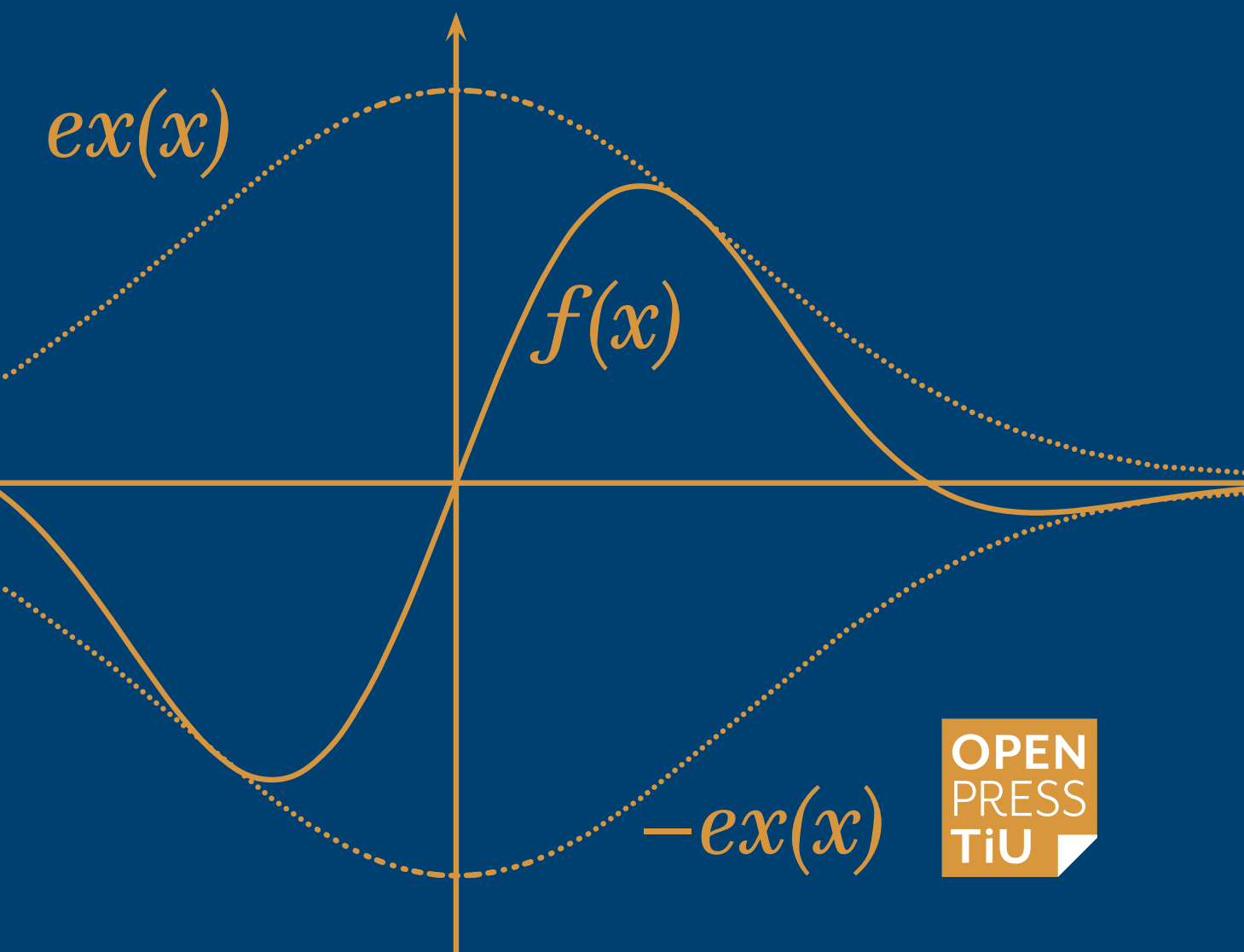


JACOB ENGWERDA

Introduction to **Optimization** **for Economists**



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INTRODUCTION TO OPTIMIZATION FOR ECONOMISTS

Introduction to Optimization for Economists

Jacob Engwerda

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Prologue

This book originates from a crash-course *refresher in mathematics* that I taught for many years at Tilburg University to research master and beginning PhD students in economics, prior to my retirement in 2019. After my retirement, the lecture notes from this course were used as the foundation for writing this book. Additional theory was incorporated, and to make the material accessible at different levels, numerous exercises were added. All exercises are worked out in the accompanying volume *Exercise Solutions for An Introduction to Optimization for Economists*.

A disclaimer is in order. Many of the exercises have not yet been tested by students. Although I have made every effort to work them out carefully, typographical errors or other inaccuracies may still be present. Readers are therefore encouraged to report any errors on the website mentioned below, so that future readers may benefit from these corrections.

My personal hope is that students around the world will enhance their understanding through the mathematical approach to optimization presented in this book. In particular, I believe the book can serve as a useful stepping stone towards the study of more specialized areas in optimization.

I would like to thank Tilburg University, and especially Tilburg University Press, for making it possible to publish this book as open access. I am particularly grateful to Beatriz Lourenço Barrocas Neves Ferreira for her support throughout the publishing process.

Last but not least, I would like to thank my wife, Carine, for giving me the space and freedom to devote so much of my time writing this book and conducting research.

If you have any comments on the book, you are most welcome to post and share them on the DOI-site: <https://doi.org/10.56675/9789403860336>.

Writing this book has been a rewarding way to spend part of my retirement. If you feel inclined to show your appreciation, you may consider a small donation to the bank account below. I will use this money then to make a hiking tour with Carine. You may use next Wise bank account for that purpose: Jacob Christiaan Engwerda, IBAN: BE21967500746803, Belgium, Swift/BIC: TRWIBEB1XXX.

Hoping you may benefit from the book, Jacob Engwerda.

Preface

Economics is the social science that studies the production, distribution, and consumption of goods and services. A central concern of the discipline is how economic agents behave and interact, and how economies function. In line with this focus, a fundamental distinction in economics is made between microeconomics and macroeconomics.

Microeconomics examines the behavior of individual components of the economy, such as households and firms, buyers and sellers, as well as markets and their interactions. Macroeconomics, by contrast, analyzes the economy as a whole and addresses issues such as unemployment, inflation, economic growth, and monetary and fiscal policy.

To support such analyses, economists often employ mathematical tools to model specific situations. By using mathematical abstractions of real-world phenomena, one aims to gain deeper insight into the underlying mechanisms at work. This, in turn, may lead to a better understanding of agent behavior and a more efficient allocation of scarce resources.

In this book, we review a number of standard mathematical tools developed to determine optima of functions. Consequently, the focus lies on the second aspect of the analysis described above: given a specific mathematical model of an economic problem, how can one derive conclusions that - provided the model adequately captures the key features of the problem - contribute to a better allocation of resources?

Starting from the concept of a function, we present conditions under which optima can be derived. Both single-variable and multi-variable functions are considered. Since resources are typically scarce, we then study optimization problems in which the domain of the function is restricted, giving rise to so-called constrained optimization problems. A particularly important subclass of these problems consists of convex optimization problems. Due to their favorable numerical properties - especially in the case of linear programming - they are widely used in applications and are therefore treated in a separate section.

Finally, we consider optimization problems in which outcomes are affected by the actions of more than one agent. This line of research is known as game theory. In the final section of the book, we introduce and study several commonly used concepts from this field.

This textbook is intended for students who wish to acquire mathematical techniques that they can later apply to their own economic problems. The main text is accessible to students of economics as well as those pursuing more technically oriented fields of study.

Mathematical technicalities required for full theoretical rigor are largely omitted from the main text. The objective is to provide students with a clear understanding of the underlying ideas, an intuitive sense of the conditions under which particular mathematical tools can be applied, as well as guidance on how to use them.

Each chapter concludes with a set of exercises, which are divided into three categories. The first category consists of exercises that allow students to test their mastery of the material presented. The

second category encourages deeper engagement with mathematical rigor, in particular through proofs of theorems introduced in the main text and related topics. The third category how the presented theory can be applied in economics to gain better insight into real-world phenomena.

All exercises are fully worked out in a separate accompanying volume entitled *Exercise Solutions for An Introduction to Optimization for Economists*.

Finally, the book concludes with a short bibliography, providing students with references to additional literature and related topics.

Notation and Symbols

$f(x) = O(x)$ as $x \rightarrow 0$	there exist c, ϵ such that $\ f(x)\ \leq c\ x\ $ when $0 < \ x\ \leq \epsilon$.
$\mathbb{R}$	Real numbers
$\mathbb{R}_+$	strictly positive real numbers
$\mathbb{R}^n$	set of vectors with n real entries
$\mathbb{R}^{n \times m}$	set of $n \times m$ matrices with real entries
$f^{-1}(x)$	inverse function of $f(x)$
$f^{(n)}(x)$	n^{th} order derivative of $f(x)$
x^T	transpose of the vector x
$\ x\ $	length $\sqrt{x^T x}$ of vector x , or, distance vector x to the origin
$\frac{\partial f}{\partial x_i}$, or, $D_i f(x)$, or $f'_{x_i}(x)$	partial derivative of f w.r.t. x_i
$D_u f(x)$	directional derivative $f(x)$ in the direction u
$Df(x) = [D_1 f(x) \cdots D_n f(x)]$	row matrix of partial derivatives (or, Jacobian of $f(x)$)
$f'(x)$	the derivative of $f(x)$ (which equals $Df(x)$ if it exists)
C^1	set of differentiable functions which partial derivatives are continuous
C^k	set of functions which all k^{th} -order partial derivatives are continuous
$f''(x)$, or, Hessian $H(x)$	second order derivative of $f(x)$, i.e. $[(D'_1 f(x))^T \cdots (D'_n f(x))^T]^T$
A^T	transpose of matrix A
$A > 0$	positive definite matrix A
$A \geq 0$	positive semi-definite matrix A
$A < 0$	negative definite matrix A
$A \leq 0$	negative semi-definite matrix A
$\det A$	determinant of matrix A
A^{-1}	inverse of matrix A
$I_{n \times n}$	identity matrix in $\mathbb{R}^{n \times n}$
$T_S(a)$	tangent cone to S at a
$TL_S(a)$	set of linearized feasible directions to S at a

$\subset$	subset
$\cup$	union
$\cap$	intersection
$\forall$	for all
$\exists$	there exists
$\mathbb{R}^n/\{0\}$	all $x \in \mathbb{R}^n$ except vector 0
$x \geq y$	componentwise inequality between vectors x and y
$f_D(\lambda)$	function associated with dual problem
$\mathcal{A}$	unit simplex
$S \setminus T$	all elements from S not belonging to T
$\bar{x}_{-i}$	$\bar{x}$, except entry i which is deleted
$\bar{x}_{\setminus i}$	$\bar{x}$, except entry i which is arbitrary
$\mathcal{R}(x_L)$	best response set follower against x_L
Λ	nonnegative orthant

CHAPTER 1

Static Optimization: The Scalar Case

In the first part of this textbook, we study the conditions under which a function of one or more variables attains an optimum. We begin with a detailed analysis of optimization problems involving a single variable. The results are then extended to the multi-variable case in the subsequent chapter.

To identify optima of functions, it is necessary to introduce several preliminary concepts. First, we clarify what is meant by a function; this issue is addressed in the next subsection. In addition, we review a number of elementary functions and some of their key properties that frequently arise in the literature.

A fundamental result in optimization theory is Weierstrass' theorem, which states that a continuous function defined on a compact domain always attains both a global maximum and a global minimum. To gain intuition for this important theorem, we briefly and informally discuss compact sets and continuous functions in two separate subsections.

1.1 Functions of one variable

We begin by introducing the mathematical concept of a function of a single variable. Let $\mathbb{R}$ denote the set of real numbers.

DEFINITION 1.1 A function f from a subset D of $\mathbb{R}$ to $\mathbb{R}$ (write $f : D \rightarrow \mathbb{R}$) defines a rule which assigns to each $x \in D$ a unique element $y \in \mathbb{R}$. The element y is called the *image* or *value* of the element x and we write $y = f(x)$.

The set D is called the *domain* of f and the set $f(D) := \{f(x) \mid x \in D\}$ the *range* of f .

A point $\bar{x} \in D$ where $f(\bar{x}) = 0$ is called a *zero* of the function f .

Probably the most well-known function is the identity function that returns the value that is used as its argument, i.e., $f(x) = x$. Usually, this function is denoted as $\text{id}(x)$ (so $\text{id}(x) = x$ for all $x \in D$).

EXAMPLE 1.2 Consider $f(x) = x^2 + 1$, for x in the interval $[-1, 1]$. Since to every x in this interval the unique element $x^2 + 1$ is assigned, f defines a function on the interval $[-1, 1]$. The domain of f is the interval $[-1, 1]$ and the range of f is the interval $[1, 2]$.

Note that in this example most of the elements in the range are attained more than once. Furthermore, this function has no zero's, as the equation $x^2 + 1 = 0$ has no solution.

EXERCISE 1.1 Determine for the rules f , defined below, a set D as large as possible such that f defines a function from $D \rightarrow \mathbb{R}$. Calculate next the range of these functions.

- a. $f(x) = x + 1$. b. $f(x) = \frac{1}{x}$. c. $f(x) = \frac{1}{(x-1)^2}$. d. $f(x) = \{x \mid x^2 = y, \text{ where } y \in [1, 4]\}$.

The *graph* of a function $f(x)$ is a graphic representation of the function in a coordinate system with two axis. The horizontal axis denotes the elements of the domain and the vertical axis the corresponding function value. That is, $\{(x, f(x)) \mid x \in D\}$. Figure 1.1 visualizes the graph of f for Example 1.2.

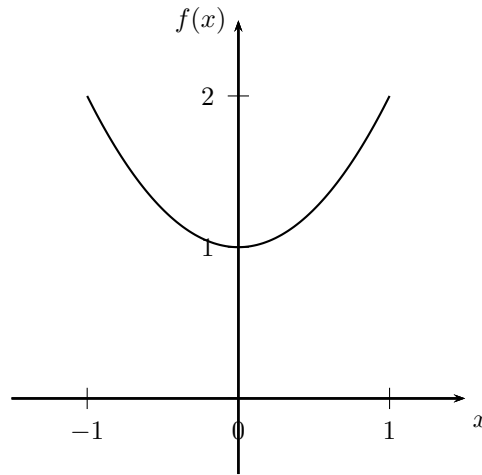


FIGURE 1.1 – Graph of $f(x) = x^2 + 1$.

The example illustrates that geometrically we have the following characterization for functions.

PROPOSITION 1.3 (*Vertical line test*)

A rule $f : D \rightarrow \mathbb{R}$ defines a function if and only if each vertical line drawn through a point of the domain D meets the graph of f in exactly one point.

In case, geometrically, also each horizontal line drawn through a point y of the range of f meets the graph of f in exactly one point the function is called invertible (or stated in terms of the previous proposition: a horizontal line test applies). The function that expresses the variable x in terms of the variable y is called the inverse function and denoted by $f^{-1}(y)$. More formal,

DEFINITION 1.4 Let $f : D \rightarrow \mathbb{R}$ define a function on D with range R . Then, if with each element $f(x) \in R$ there corresponds a unique element $x \in D$ the function f is called *invertible*. Or, stated differently, there exists a function $f^{-1}(y)$ from R to D such that $f(f^{-1}(y)) = y$ for all $y \in R$ and $f^{-1}(f(x)) = x$ for all $x \in D$. $f^{-1}(y)$ is called then the inverse function of f .

In Exercise 1.30, you are asked to show that if a function is invertible, then it has exactly one inverse function.

EXAMPLE 1.5 Consider the function $f(x) = x^2 + 1$, defined on the interval $[0, 1]$. From Figure 1.1, it is evident that f is invertible on this interval. To determine the inverse function analytically, we solve for x in terms of y from the equation $y = x^2 + 1$. This yields $x^2 = y - 1$. Since x is restricted to be nonnegative on $[0, 1]$, it follows that $x = \sqrt{y - 1}$ is the specification we are looking for. Therefore, the inverse function is $f^{-1}(y) = \sqrt{y - 1}$. Replacing the symbol y by the symbol x , we obtain $f^{-1}(x) = \sqrt{x - 1}$.

EXERCISE 1.2 Determine which of the functions f , defined below, is invertible. For all functions that are invertible, determine the inverse function f^{-1} .

- a. $f(x) = x + 1$. b. $f(x) = x^2$, $x \in [-1, 1]$. c. $f(x) = -x^2 + 1$, $x \in [0, 1]$.

Note that geometrically the inverse function is obtained from f by plotting the same graph but now with the y -axis as the horizontal axis and the x -axis as vertical axis. Geometrically this manipulation can be obtained by a reflection of the graph of f through the line $y = x$. Figure 1.2 illustrates this for Example 1.5.

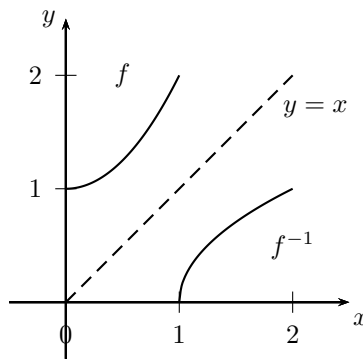


FIGURE 1.2 – Graph of $f(x) = x^2 + 1$ and it's inverse on $[0, 1]$.

We list some frequently occurring functions below:

A. Affine functions: $f(x) = ax + b$, where a, b are scalars.

Geometrically, an affine function represents a straight line, where b is the *intercept* on the y -axis and a is the *slope of the line* (see Figure 1.3).

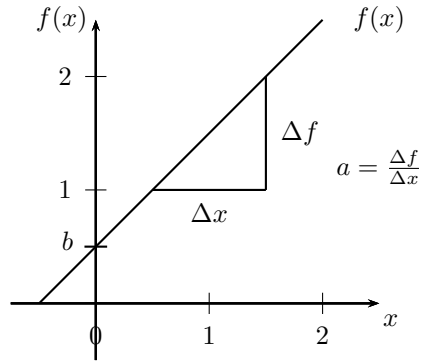


FIGURE 1.3 – Affine function $f(x) = ax + b$.

B. Polynomial functions: $f(x) = a_0 + a_1x + \dots + a_nx^n$, where a_i are scalars.

Here n is called the *degree of the polynomial* (assuming that $a_n \neq 0$). Special cases are $n = 0$, the *constant function*, and $n = 1$, the *affine function*. The equation $f(x) = 0$ has at most n (real) solutions. Or, stated differently, the function has at most n *zero's*. Figure 1.4 visualizes a typical plot for a third order degree polynomial.

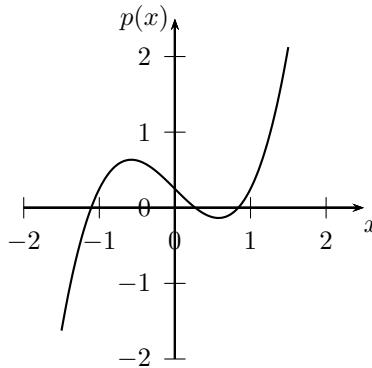


FIGURE 1.4 – Graph of a third order polynomial $p(x)$.

C. Rational functions: Suppose $p(x)$ and $q(x)$ are polynomial functions. Then $f(x) = \frac{p(x)}{q(x)}$, for all x where $q(x) \neq 0$, is called a rational (ratio of polynomials) function.

Typical for the graph is that at those points where $q(x) = 0$ usually $f(x)$ attains either arbitrarily large values (in which case we say $f(x)$ diverges to ∞ (infinity)) or either arbitrarily small values (in which case we say $f(x)$ diverges to $-\infty$). Figure 1.5 illustrates a typical graph.

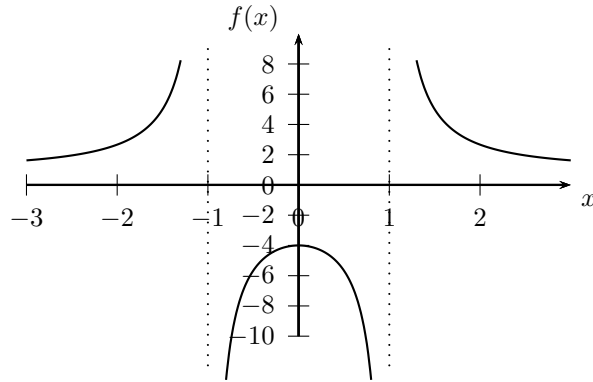


FIGURE 1.5 – Graph of the rational function $f(x) = \frac{x^2+4}{x^2-1}$ for $x \neq \pm 1$.

D. Exponential functions: $f(x) = ma^x$, where a is a positive number and m is some scalar.

The number a is usually referred to as the *base*.

As an example, consider compound interest. Suppose an amount m dollars is invested in a bank that pays an annual compound interest rate of r . The total amount after one year is

$$y(1) = m + mr = m(1 + r).$$

After two years, the accumulated amount is

$$y(2) = m(1 + r) + [m(1 + r)]r = m(1 + r)^2.$$

In general, after x years the accumulated amount is

$$y(x) = m(1 + r)^x.$$

Thus, $y(x)$ is an exponential function with base $(1 + r)$. Figure 1.6 illustrates the graph of an exponential function with base 2.

A particularly important base in the study of exponential functions is Euler's number

$$e = 2.718....$$

This constant arises, for instance, as the limit obtained when a bank pays an interest $r = 1$, and compounds the interest n times per year, with each payment equal to a fraction $\frac{1}{n}$, as n tends to infinity. Formally,

$$e = \left(1 + \frac{1}{n}\right)^n, \text{ if } n \text{ diverges to infinity.}$$

In particular, this implies that if a bank pays interest at rate r according to this compounding scheme, the accumulated amount after one year is e^r . So, when an exponential function of the form $f(x) = e^x$ appears in the literature, the symbol e refers to this constant.

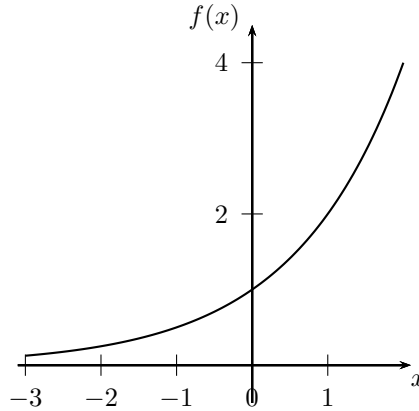


FIGURE 1.6 – Graph of the exponential function $f(x) = 2^x$.

E. Logarithmic functions: $f(x) = {}^a\log(x)$, where a is a positive number (the *base* of the logarithm)¹. Using the horizontal line test for the exponential function $y = a^x$, it is clear that this function is invertible. The inverse of this exponential function is called the logarithmic function and denoted by ${}^a\log(y)$. So, by definition

$$x = {}^a\log(y) \text{ if and only if } y = a^x.$$

Since the range of $f(x) = a^x$ is the set $\{y \mid y > 0\}$, this relationship implies that the logarithmic function is defined only for all strict positive scalars.

Special notations are attached to the inverse functions corresponding with the basis 10, Euler's number e , and 2. The logarithm to base $a = 10$ is called the *common logarithm* ($\log(x)$) and has many applications in science and engineering. The *natural logarithm* ($\ln(x)$) has the constant e as its base; its use is widespread in pure mathematics, especially calculus, and economics. The *binary logarithm* uses base $a = 2$ and is prominent in computer science.

Since the graph of the inverse function is obtained by reflecting the graph of the original across the line $y = x$, a typical graph of a logarithmic function appears as shown in Figure 1.7.

A number of elementary properties which immediately follow from the definition of the logarithmic function $f(x) = {}^a\log(x)$ are:

1. $f(0) = 1$.
2. The function is an increasing function (i.e., it rises to the right) for all $a > 1$.
3. The function is a decreasing function (i.e., it falls to the right) for all $0 < a < 1$.

¹ Another notation frequently used in literature is $f(x) = \log_a(x)$.

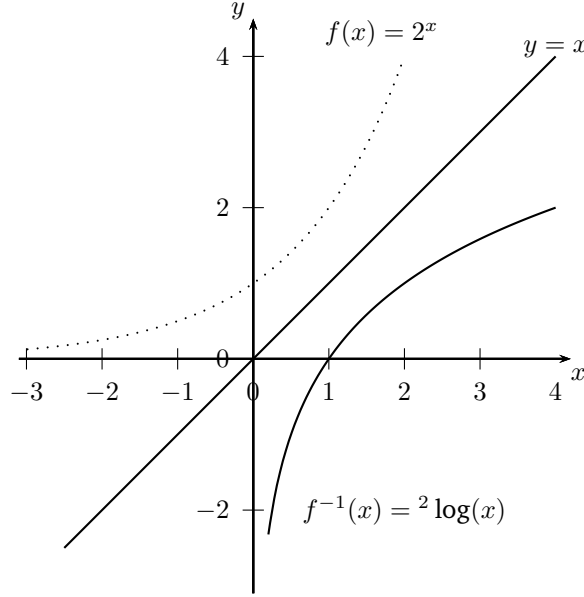


FIGURE 1.7 – Graph of the logarithmic function ${}^2\log(x)$.

4. The function is negative when $0 < x < 1$ and $a > 1$.
5. The function is positive when $x > 1$.
6. The function is not defined when x is negative.

Similarly, the following elementary rules for calculation with logarithms result immediately from the rules for calculation with exponents:

Calculation rules for Exponents	Calculation rules for Logarithms
$a^x a^z = a^{x+z}$	${}^a\log(uv) = {}^a\log(u) + {}^a\log(v)$
$\frac{a^x}{a^z} = a^{x-z}$	${}^a\log\left(\frac{u}{v}\right) = {}^a\log(u) - {}^a\log(v)$
$(a^x)^z = a^{xz}$	${}^a\log(u^n) = n * {}^a\log(u)$
$a^x = a^y$ implies $x = y$	${}^a\log(x) = {}^a\log(y)$ implies $x = y$

F. Combining functions:

Let f and g be two functions from a set D to $\mathbb{R}$. The *sum* of f and g , denoted by $f + g$ is the function from D to $\mathbb{R}$ defined by

$$(f + g)(x) := f(x) + g(x) \text{ for all } x \in D.$$

Similarly, for any real number λ , the function λf is defined by

$$(\lambda f)(x) := \lambda f(x) \text{ for all } x \in D.$$

More generally, the *product* and *quotient* of functions are defined by

$$fg(x) := f(x)g(x) \text{ and } \left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}, \text{ whenever } g(x) \neq 0.$$

Finally, the *composition* of functions is defined as follows. Let: $g : S \rightarrow T$ and $f : T \rightarrow \mathbb{R}$. Then the composite function $f \circ g$ is given by

$$(f \circ g)(x) := f(g(x)) \text{ for all } x \in S.$$

EXAMPLE 1.6 Let $f(x) = x + 2$ and $g(x) = x^2 + 1$. Then:

$$(f + g)(x) = x^2 + x + 3; (fg)(x) = (x + 2)(x^2 + 1); \left(\frac{f}{g}\right)(x) = \frac{x+2}{x^2+1}; (f \circ g)(x) = x^2 + 3 \text{ and } (g \circ f)(x) = (x + 2)^2 + 1.$$

EXERCISE 1.3

- a. Simplify $\frac{^a \log(x)}{^a \log(y) + ^a \log(z)}$.
 b. Simplify $e^{2 \ln(x)} - \ln(e^{2x})$.
 c. Solve $2^{x^2} = 4^{-3x-4}$.

d. Solve $\left(\frac{1}{4}\right)^{x^2} = 64(2^{-2x})^4$.

e. Solve $2 \ln(x - \frac{1}{2}e) - \ln(x - \frac{3}{4}e) = 1$.

EXERCISE 1.4 Let $f(x) = x^2 + 1$ and $g(x) = \log(x^2 + 1)$. Determine

- a. $(f + g)(x)$. b. $(fg)(x)$. c. $\left(\frac{f}{g}\right)(x)$. d. $(f \circ g)(x)$. e. $(g \circ f)(x)$.

1.2 Compact Sets

To introduce the notion of compactness of a set $S \subset \mathbb{R}$ we first introduce the notions of interval and bounded sets.

DEFINITION 1.7 An *interval* I is a set of real numbers with the property that with every two numbers $x, y \in I$ also every number between them belongs to I .

DEFINITION 1.8 A set $S \subset \mathbb{R}$ is called *bounded* if there exists a number m such that for all $x \in S$ it holds: $|x| < m$.

In describing intervals we use the following notation:

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}; [a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}; [a, b) := \{x \in \mathbb{R} \mid a \leq x < b\};$$

$$(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}.$$

These are all bounded intervals. Figure 1.8, panel a, illustrates the bounded interval $[1, 2]$. We will also consider unbounded intervals. For these we use the notation:

$$\begin{aligned}(a, \infty) &:= \{x \in \mathbb{R} \mid x > a\}; & [a, \infty) &:= \{x \in \mathbb{R} \mid x \geq a\}; & (-\infty, b) &:= \{x \in \mathbb{R} \mid x < b\}; \\ (-\infty, b] &:= \{x \in \mathbb{R} \mid x \leq b\}.\end{aligned}$$

Next we introduce the notion of openness and closedness of a set S . To that end first recall that the *complement set* of a set S are all numbers that don't belong to S , i.e. $\{x \in \mathbb{R} \mid x \notin S\}$. So, for example, if $S = \{1, 2\}$ then the complement set of S are all numbers in $\mathbb{R}$ different from 1 and 2.

DEFINITION 1.9 A set S of real numbers is called *open* if for every $x \in S$ there exists an interval (a_x, b_x) containing x , such that all elements of this interval belong to S .

The set S is called *closed* if the complement set of S is open. Or, equivalently², if one considers an arbitrary sequence of numbers $\{s_n\}_{n=1}^{\infty}$, all belonging to S , that converges to some limit, s , then this limit s also belongs to S .

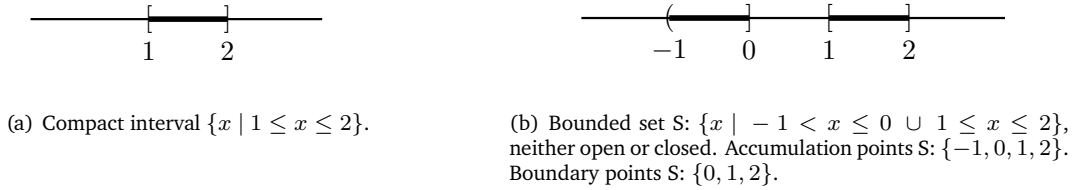


FIGURE 1.8 – Sets and properties.

With this definition the intervals (a, b) , (a, ∞) and $(-\infty, b)$ are open, the intervals $[a, b]$, $[a, \infty)$ and $(-\infty, b]$ are closed, the intervals $[a, b)$ and $(a, b]$ are neither open or closed and the set of all real numbers, $\mathbb{R}$, is as well open as closed (see Exercise 1.28).

A simple yet often useful property of a finite collection of open (or closed) sets is that both their union and intersection share the same property. We formalize this property below.

PROPOSITION 1.10 If S and T are two open sets, then both $S \cap T$ and $S \cup T$ are open as well. Similarly, if S and T are closed sets, then $S \cap T$ and $S \cup T$ are also closed.

More generally, the intersection of an arbitrary (possibly infinite) collection of closed sets is closed, and the union of an arbitrary (possibly infinite) collection of open sets is open.

Using the above definitions we can define finally the notion of a compact set.

² This can be proved formally, but is not trivial to show.

DEFINITION 1.11 A set S of real numbers is called *compact* if it is both bounded and closed.

It is easy to verify that in Figure 1.8, panel (a), the interval $[1, 2]$ is compact.

The set S in panel (b), which is the union of the intervals $(-1, 0]$ and $[1, 2]$, is bounded but not compact, since S is not closed.

To see this formally, consider the sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_n = -1 + \frac{1}{n}$. This sequence converges to -1 . Notice that $s_n \in S$ for all $n = 1, 2, \dots$. Therefore, -1 is an accumulation point of S , yet $-1 \notin S$.

EXAMPLE 1.12 Consider $S = (1, 2] \cup [3, 4]$. Then S is bounded, not open, not closed and not compact.

Consider $S = [1, 2] \cup [3, 4]$. Then S is bounded, not open, closed and compact.

Consider $S = [1, 2] \cup [3, \infty)$. Then S is not bounded, not open, closed and not compact.

Consider $S = \{1\}$. Then S is bounded, not open, closed and compact.

Another intuitive way to formalize the concept of a set being closed is by examining its boundary and its accumulation points.

DEFINITION 1.13 A point $b \in S$ is called a *boundary point* of the set S if every interval (b_l, b_r) , containing b , contains at least one point not belonging to S . Or, equivalently, if there exists a sequence of numbers, all not belonging to S , that converges to b .

A point a is called an *accumulation point* of the set S if every interval (a_l, a_r) containing a , contains at least one point in S different from a and at least one point not in S .

Equivalently, a is an accumulation of S if there exists a sequence of points not in S that converges to a , and another sequence of points in S (with all terms distinct from a beyond a certain index) that also converges to a .

Notice the subtle difference between an accumulation point and boundary point of a set S , that the first-mentioned one does not necessarily have to belong to the set, whereas a boundary point by definition always belongs to the set S .

EXAMPLE 1.14 Consider the interval $I = [0, 1)$. I has one boundary point: 0. The accumulation points of I are 0 and 1.

By definition 1.9, we conclude that a set S is closed if and only if it contains all of its accumulation points. Or, stated differently, all its accumulation points are boundary points.

EXERCISE 1.5 Determine the boundary and accumulation points of next sets. Motivate which of these sets is bounded, open, closed or compact, respectively.

- a. $[1, 2] \cup [3, \infty)$. b. $\{1, 2\}$. c. $[1, 2) \cup (2, 3]$. d. $[1, 3) \cap (0, 2]$.

1.3 Continuous Functions

In this section we introduce in an informal way the notion of continuity of a function. A formal set-up is skipped. Readers interested in a more formal approach of this notion can find some more background on this in the exercises (see, e.g., Exercise 1.32). Otherwise, for a more extensive treatment, they should consult the literature (see, e.g., [6][Section 6.4]). This section also provides some important properties of continuous functions and we argue that most functions encountered in literature are continuous.

Intuitively, a function is called continuous at a point x_0 if its graph has no "jump" or "break" above x_0 . In other words, the graph is connected within an open interval containing x_0 , so that it can be drawn without lifting the pencil from the paper.

Equivalently, a function f is continuous at x_0 if, for any sequence $\{s_n\}_{n=1}^{\infty}$ in the domain that converges to x_0 , the corresponding sequence of function values $\{f(s_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

DEFINITION 1.15 Let the function f be defined on an open interval I and $x_0 \in I$. Then f is called *continuous at $x = x_0$* , in case for every sequence $\{s_n\}_{n=1}^{\infty}$ that approaches x_0 it holds that $\{f(s_n)\}_{n=1}^{\infty}$ approaches $f(x_0)$. If f is continuous at every $x_0 \in I$, f is called *continuous on I* . If f is not continuous it is called *discontinuous*.

Clearly this definition can be straightforwardly generalized to define continuity of a function that is defined on a set S that is the union of open intervals that have an empty intersection. Figure 1.9 illustrates the basic idea of continuity. Moreover, Figure 1.5 illustrates a continuous function which is defined on a set which is the union of disjunct intervals.

In the sequel we will make the convention that if we talk about continuity of a function on a set S , S is the union of open intervals that have an empty intersection.

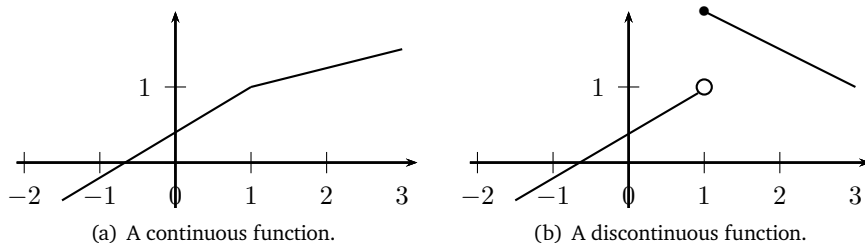


FIGURE 1.9 – Continuity of functions.

Using this formalization it can be easily shown that next theorem applies.

THEOREM 1.16 *Let f and g be continuous functions on a set S , and λ, μ be real numbers. Then the following functions are continuous on S too:*

$$i) \lambda f + \mu g; ii) fg; iii) \frac{f}{g}, \text{ provided that } g(x) \neq 0 \text{ for all } x \in S.$$

Furthermore, if f is continuous on the range of $g(S)$ and g is continuous on S , also the composite function $f \circ g$ is continuous on S .

Clearly the constant functions and the linear function $f(x) = x$ are continuous on $\mathbb{R}$ (see also Exercise 1.32). Using repeatedly Theorem 1.16, items i) and ii), shows then that every polynomial function is continuous too. Consequently, using item iii), it follows that also all rational functions are continuous on all intervals where they are defined. It can also be shown that the exponential functions are continuous on $\mathbb{R}$ and the logarithmic functions, for $x > 0$, too. So, by combining all these functions using addition, multiplication, division (except by zero), and composition we conclude from Theorem 1.16 that such functions are continuous at all points where they are defined.

We now come to a result which may seem so obvious as to be hardly worth mentioning. It is, however, of fundamental importance and serves as a foundation stone in the development of calculus.

PROPOSITION 1.17 *(Continuity property)*

Let the continuous function f be defined on an interval I . Then,

- 1. the set $f(I)$ ³ is also an interval;*
- 2. if I is a compact interval $f(I)$ is also a compact interval.*

Figure 1.10 visualizes Proposition 1.17. One important corollary of this result is the next so-called Intermediate Value Theorem.

THEOREM 1.18 *(Intermediate Value Theorem)*

Let the continuous function f be defined on an interval I . Assume $x_i \in I$, $i = 1, 2$, with $x_1 < x_2$. Then, if y lies between $f(x_1)$ and $f(x_2)$ there exists an x_3 between x_1 and x_2 such that $f(x_3) = y$.

From Proposition 1.17 one can also derive immediately next important theorem in optimization.

THEOREM 1.19 *(Weierstrass Theorem)*

Let f be continuous on a compact set S . Then, f attains both a maximum value and a minimum value on S .

³ formal this set is defined as $\{f(x) \mid x \in I\}$.

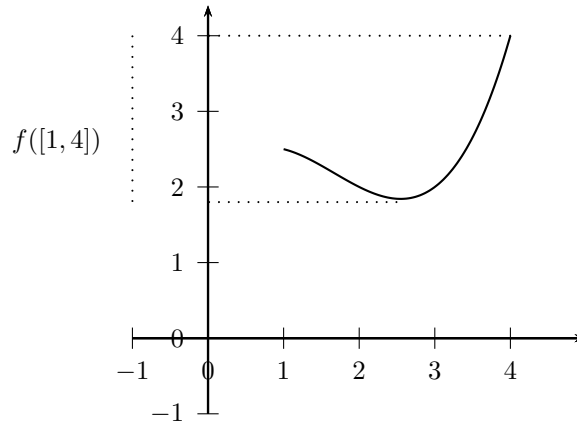


FIGURE 1.10 – Range of continuous function of compact set.

EXAMPLE 1.20 Consider the function $f(x) = x^2 - xe^{x^2-4x+1} + \ln(x^2 + 1)$ on $[0, 1]$. From Theorem 1.16 it follows straightforwardly that f is continuous on $[0, 1]$. Furthermore, $[0, 1]$ is a compact set. So by Weierstrass' theorem, f attains both a maximum and minimum on $[0, 1]$. Notice (as we will see later on) that it is analytically not possible to calculate the precise locations where this maximum and minimum are attained.

EXERCISE 1.6 Motivate which of the next functions is continuous.

- a. $f(x) = \begin{cases} \log(x^2 + 1), & x \leq 0 \\ x, & x > 0 \end{cases}$. c. $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.
- b. $f(x) = xe^{x^2 + \ln(x^2 + 1)}$.

EXERCISE 1.7 Motivate whether the next functions have an optimum.

- a. $f(x) = \frac{1}{x}, x > 0$. b. $f(x) = \frac{x^2 - e^x}{\ln(x^2 + 2)}, -10 \leq x \leq 12$.

1.4 Differentiable Functions

The rate at which a function changes - i.e., the change in $f(x)$ per unit change in x - is a fundamental concept. It allows us to approximate how much the value of a function will increase or decrease when x changes by one unit. In economics, this rate of change is referred to as the *marginal value*. For instance, the marginal production cost measures how total production costs change when an input factor is increased by one unit.

At a given point x_0 , this rate of change can be approximated by the so-called derivative of the function at x_0 , provided the function behaves smoothly. At points where a function reaches a maximum or a minimum, this rate of change equals zero. Consequently, derivatives play a central role in solving

optimization problems. By determining all points at which the derivative equals zero, one obtains an initial set of candidate points where an optimum may occur.

For the elementary functions introduced earlier, in fact, derivatives can be computed analytically. By assigning to every point in the domain of the function its corresponding derivative value, we obtain a function again, the so-called derivative function. So, whenever a function behaves smoothly, solving an optimization problem essentially reduces to finding the zeros of the derivative function.

This leads to the question of what it means for a function to be *smooth*. From a geometric perspective, a function f is said to be smooth at a point x_0 if there exists a unique line that just "touches" the graph of f at that point. When this is the case, function values near x_0 can be accurately approximated using this line, allowing us to reliably estimate the effect of small changes in x on the function value. This idea is illustrated in Figure 1.11. Informal, a line that just "touches" the graph of f at the point x_0 is called a *tangent line* to the graph of f at x_0 .

So, the question is therefore under which conditions there will be a unique tangent line to the graph of f at x_0 . To answer that question we consider the average change in the function value if the argument of the function changes from x_0 to $x_0 + \Delta x$. This change is represented by the so-called *difference quotient*:

$$\frac{\Delta f(x_0)}{\Delta x} := \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

For example consider $f(x) = 3x^2 - 4$. Then the average change in f that occurs at x_0 if x changes into $x_0 + \Delta x$ is given by evaluating f at x_0 and at $x_0 + \Delta x$, respectively. Notice that $f(x_0) = 3x_0^2 - 4$ and $f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4$. Therefore, the average change is

$$\frac{\Delta f(x_0)}{\Delta x} = \frac{3(x_0 + \Delta x)^2 - 4 - [x_0^2 - 4]}{\Delta x} = \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x} = 6x_0 + 3\Delta x.$$

So, if $x_0 = 3$ and $\Delta x = 4$, the average change of f is $6 * 3 + 3 * 4 = 30$. This means that, on the average, as x changes from 3 to 7, the change in f is 30 units per unit change in x .

Figure 1.11 visualizes this difference quotient. From the graph we infer that geometrically the difference quotient measures the slope of the chord PQ , i.e. $\frac{\Delta f(x_0)}{\Delta x} = \frac{QR}{PR}$. Furthermore we observe that if Δx goes to zero, the slope of the chord PQ approaches the slope of the line m , which is tangent to $f(x)$ in the point x_0 . Note that in this graph we took Δx positive, but that this property also holds if we would have chosen Δx negative. That is, if we would have approached $f(x_0)$ from the left.

By letting Δx going to zero in the difference quotient $\frac{\Delta f(x_0)}{\Delta x}$ we measure in fact the instantaneous effect a change in x_0 will have on $f(x)$. In the sequel we will call this limit value (if it exists) the *derivative* of f at the point x_0 and denote it by $f'(x_0)$. In case this derivative exists, we will call the line through $f(x_0)$ which slope equals this derivative *the tangent line* of f at x_0 . The equation of this line is: $m(x) = f(x_0) + f'(x_0)(x - x_0)$.

In fact, using this equation for the tangent line, the statement that the difference quotient $\frac{\Delta f(x_0)}{\Delta x}$ approaches zero as Δx approaches zero can also be reformulated in terms of our original problem

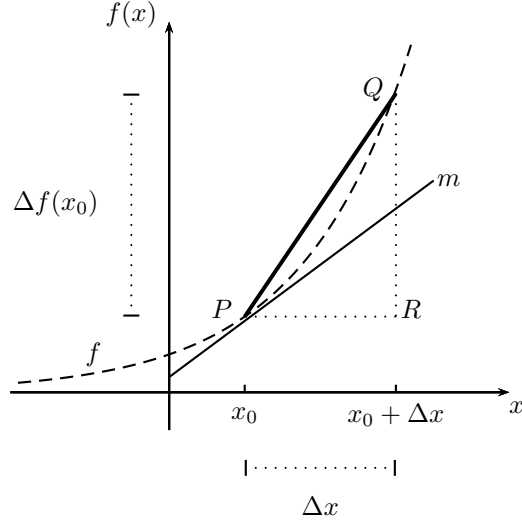


FIGURE 1.11 – Difference quotient and differentiability.

formulation. That is, notice that:

$$\begin{aligned} \frac{\Delta f(x_0)}{\Delta x} &= f'(x_0) \text{ if } \Delta x \text{ approaches zero} \Leftrightarrow \\ \frac{f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x}{\Delta x} &= 0 \text{ if } \Delta x \text{ approaches zero} \Leftrightarrow (\text{Replace } \Delta x = x - x_0) \\ \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} &= 0 \text{ if } x \text{ approaches } x_0 \Leftrightarrow \\ \frac{f(x) - m(x)}{x - x_0} &= 0 \text{ if } x \text{ approaches } x_0. \end{aligned}$$

From the last equation, we infer that existence of the derivative $f'(x_0)$ is equivalent with next statement. The distance between the function value $f(x, y)$ and the corresponding value on the tangent line, $m(x) = f(x_0) + f'(x_0)(x - x_0)$, divided by the distance of x to x_0 , goes to zero if x goes to x_0 . This formulation will be used in the next chapter to generalize the notion of "smoothness" to functions of more than one variable.

Next definition summarizes in a formal way the above discussion.

DEFINITION 1.21 Suppose f is defined on an open interval I containing the point x_0 . The function f is said to be *differentiable* at x_0 if the difference quotient

$$\frac{\Delta f(x_0)}{\Delta x}$$

has a unique limit as Δx approaches zero. This limit is called the *derivative* of f at x_0 and is denoted by $f'(x_0)$ or $\frac{df}{dx}(x_0)$.

The *tangent line* to the graph of f at x_0 is defined as the line passing through the point $(x_0, f(x_0))$

with slope $f'(x_0)$.

If f is differentiable at every point of I , then f is said to be differentiable on I .

REMARK 1.22 As already notice above, the tangent line to the graph of f is given by

$$m(x) = f(x_0) + f'(x_0)(x - x_0),$$

if f is differentiable at x_0 .

Usually, the tangent line at x_0 has the intuitive property of "touching" the graph of f at that point: that is, there exists an open interval I containing x_0 such that the tangent line does not intersect the graph of f anywhere else within I . However, this is not always the case.

This phenomenon is illustrated in Section 1.8. In Example 1.51, we consider a function f that is differentiable on $\mathbb{R}$, with $f'(0) = 0$ and $f(0) = 0$, yet has the property that, in every interval $(x_l, 0)$ with $x_l < 0$, f attains both positive and negative values. Consequently, the tangent line at $x_0 = 0$ given by $m(x) = 0$, intersects the graph of f on any open interval to the left of $x = 0$.

In the example considered above, the difference quotient for the function $f(x) = 3x^2 - 4$ at x_0 is

$$\frac{\Delta f(x_0)}{\Delta x} = 6x_0 + 3\Delta x.$$

As Δx goes to zero, this difference quotient approaches the value $6x_0$. So, the derivative of $f(x)$ at x_0 is

$$f'(x_0) = 6x_0.$$

The tangent line to the graph of f at, for instance, $x_0 = 1$ is: $y(x) = -1 + 6(x - 1)$.

From the definition it is clear that not every function is differentiable. A well-known example to illustrate this is the function $f(x) = |x|$. We plotted the graph of this function in Figure 1.12. Obviously this graph has not a unique tangent line at $x_0 = 0$. So this function is not differentiable at $x_0 = 0$. This intuition is confirmed by considering the difference quotient. $\frac{\Delta f(x_0)}{\Delta x} = -1$ if $\Delta x < 0$ and $\frac{\Delta f(x_0)}{\Delta x} = 1$ if $\Delta x > 0$. Therefore, the difference quotient does not approach a unique value if Δx approaches zero.

It can be shown from the definition 1.21 (see Exercise 1.33) that if a function is differentiable at a point x_0 the function must be continuous at this point too. Or, stated differently, a prerequisite for a function to be differentiable at a point x_0 is that it is continuous at that point. We stress this point by formulating next theorem.

THEOREM 1.23 *Let f be differentiable on an open interval containing x_0 . Then f is continuous at x_0 too.*

As with continuity, the following property can be established for differentiable functions.

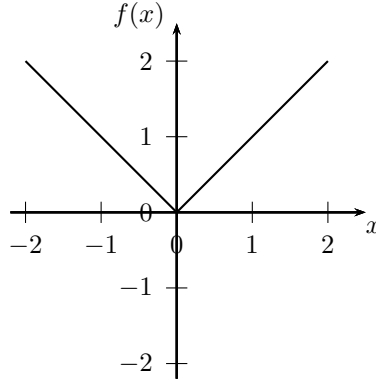


FIGURE 1.12 – $f(x) = |x|$.

THEOREM 1.24 Let f and g be differentiable on an open interval containing x_0 , and λ, μ be real numbers. Then

1. $\lambda f + \mu g$ is differentiable at x_0 , and $(\lambda f + \mu g)'(x_0) = \lambda f'(x_0) + \mu g'(x_0)$
 2. fg is differentiable at x_0 , and $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$
 3. $\frac{f}{g}$ is differentiable at x_0 (provided $g(x_0) \neq 0$), and $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{g^2(x_0)}$
- Moreover, if f is differentiable at $g(x_0)$ and g is differentiable at x_0 , also
4. $f \circ g$ is differentiable at x_0 , and $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$.

It is easily verified from Definition 1.21 that the constant function is differentiable and its derivative is zero at any point. It is also clear that the linear function $f(x) = x$ is differentiable everywhere and its derivative is 1 at any point. By a repeated use of Theorem 1.24 it follows then that every polynomial function is differentiable too and, consequently, also every rational function (at points where the denominator differs from zero). In fact one can conclude relatively easily using the above arguments that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. Geometrically it is also clear that any exponential function and logarithmic function are differentiable. This implies that most of the functions that occur in literature are differentiable.

In Table 1, below, we list the derivatives of some elementary functions.

Notice from this table that the derivative of an elementary function is an elementary function again. This implies that most of the derivatives of the functions that occur in literature are differentiable again, etc. Or, stated differently, most of the functions can be differentiated as often as we want.

EXAMPLE 1.25 From Theorem 1.24 and Table 1, it is immediately clear that the following two functions are differentiable. Moreover, their corresponding derivatives follow directly.

1. If $f(x) = \frac{1}{x} + x^2 \ln(x)$, $x > 0$, $f'(x) = -\frac{1}{x^2} + 2x \ln(x) + x^2 \frac{1}{x} = -\frac{1}{x^2} + 2x \ln(x) + x$.
2. If $f(x) = xe^{x^2+1}$, $f'(x) = e^{x^2+1} + xe^{x^2+1}2x = (1 + 2x^2)e^{x^2+1}$.

$f(x)$	$f'(x)$
c ($c \in \mathbb{R}$)	0
x^n ($n \neq 0$)	nx^{n-1}
e^x	e^x
a^x ($a > 0$)	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$a \log(x)$ ($a > 0, a \neq 1$)	$\frac{1}{x} \frac{1}{\ln(a)}$

TABLE 1.1 – Derivatives of some elementary functions

EXERCISE 1.8 Determine the derivative of the next functions $f(x)$.

- a. $f(x) = x^2 + 2 \log(x)$, $x > 0$. c. $f(x) = \frac{\ln(x)}{x+1}$, $x > 0$. e. $f(x) = \log(x^2+1) + \sqrt{x^4+1}$.
b. $f(x) = \frac{2}{x^5} + 2^x$, $x \neq 0$. d. $f(x) = x^2 e^{x^2} + 1$.

EXERCISE 1.9 Motivate which of the next functions is differentiable.

- a. $f(x) = \frac{x^2 + e^x}{\ln(x^2+2)}$. c. $f(x) = x + |x|$. e. $f(x) = x|x| + \log(e^x + 4)$.
b. $f(x) = \begin{cases} x, & x \leq 0 \\ x+1, & x > 0 \end{cases}$. d. $f(x) = \begin{cases} x, & x \leq 0 \\ 2x, & x > 0 \end{cases}$.

1.5 First-Order Condition for an Optimum of a Scalar Function

A company aims to maximize its profit. A production unit aims to minimize its cost for a given level of output. An individual aims to maximize his/her utility when choosing among different leisure options. All of these examples demonstrate that optimization problems are common in economics.

If the optimization problem has been formulated as a mathematical model involving the optimization of an objective function with respect to a single variable, it can be solved using calculus. To find the optimum, one can proceed in two steps:

ALGORITHM 1.26 To find the global optima of a function one should, basically, proceed as follows.

Step 1. Locate all local maxima and minima of the objective function.

Step 2. Compare the function value at the local maxima and minima and at the boundaries of the domain. The highest (lowest) value yields then the global maximum (minimum) of the objective function.

To accomplish Step 1, we first have to define what we mean by a local maximum (minimum). Figure 1.13 illustrates the situation.

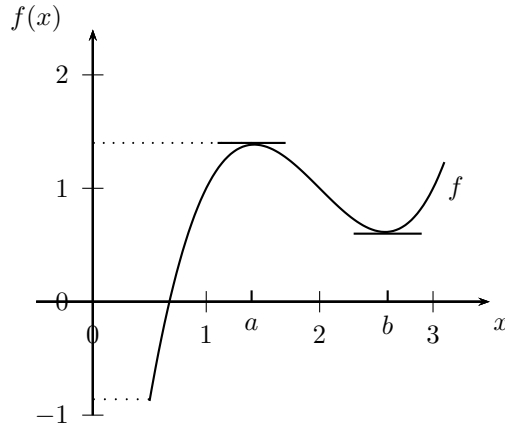


FIGURE 1.13 – Optima of a function.

The function f is said to have a local maximum at the point $x = a$ because $f(a) \geq f(x)$ for all x sufficiently close to a . Similarly, f is said to have a local minimum at $x = b$ because $f(b) \leq f(x)$ for all x sufficiently close to b . More formal we have:

DEFINITION 1.27 Let f be defined on an open interval I and let $a, b \in I$. Then we say that f has

- a *local maximum* at a if $f(x) \leq f(a)$ for all values of x in some open interval (a_l, a_r) that contains a .
- a *local minimum* at b if $f(x) \geq f(b)$ for all values of x in some open interval (b_l, b_r) that contains b .
- a *global maximum* at a if $f(x) \leq f(a)$ for all $x \in I$.
- a *global minimum* at b if $f(x) \geq f(b)$ for all $x \in I$.

Points a and b at which the function either attains a maximum or minimum value are called *extremal points*.

EXAMPLE 1.28 The function f in Figure 1.13 is considered on the interval $[0.5, 3.1]$. It has a local minimum at $x = b$; a local maximum at $x = a$; a boundary maximum at $x = 3.1$; a boundary minimum at $x = 0.5$; a global minimum at $x = 0.5$ and a global maximum at $x = a$.

As can be seen from Figure 1.13, an important property of local extrema is that the tangent line to the graph at an extremal point is horizontal. That is (see also Exercise 1.34):

THEOREM 1.29 (first-order condition) Suppose f is differentiable on the open interval I and that $a \in I$. If f has a maximum or a minimum at a , $f'(a) = 0$.

This property will be used in the following to identify an initial set of points at which a function f may attain its optimum. The requirement that the derivative of the function equals zero at an optimum is commonly referred to in the literature as the *First-Order Condition* (FOC). Since the points at which the

derivative of f becomes zero play a fundamental role in determining the optimal value of a function, we assign them a special name.

DEFINITION 1.30 Assume f is differentiable on the open interval I and that $a \in I$. A point a at which $f'(a) = 0$ is called a *stationary point* of f .

In Figure 1.14 we plotted the graph of $f(x) = x^3$. The graph shows that, though f has a stationary point at $x = 0$, this is neither a minimum nor a maximum location. In other words, stationary points of a function are not always optima. Stationary points which are neither a maximum or a minimum location are called *saddle points*. The stationary point visualized in Figure 1.14 illustrates the geometric picture behind this definition. The graph shows that there is a direction (i.e. to the left of $x = 0$) where all function values are smaller than $f(0)$ in a neighborhood of $x = 0$, and there is another direction (i.e. to the right of $x = 0$) where all function values are larger than $f(0)$ in a neighborhood of $x = 0$. In most cases this intuitive picture is correct. However, there are cases where this geometric intuition breaks down. We will comment on this in Section 1.8, below.

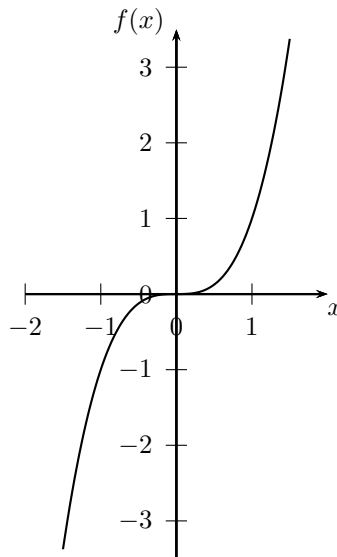


FIGURE 1.14 – $f(x) = x^3$.

As already mentioned above, to find an optimum of a function, usually, we first try to find the stationary points of the function. We illustrate this in the next example.

EXAMPLE 1.31

1. Consider $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 4$. The stationary points of f are obtained by solving $f'(x) = 0$, i.e., $x^2 - 3x + 2 = 0$. This equation has two solutions, $x = 1$ and $x = 2$.
2. Consider $f(x) = x \ln(x)$ for $x > 0$. Then the stationary points of f are found by solving $f'(x) = 0$, i.e., $\ln(x) + 1 = 0$. This equation has the unique solution $x = \frac{1}{e}$.

EXERCISE 1.10 Determine the stationary points of the next functions.

- a. $f(x) = xe^x$. b. $f(x) = \frac{x}{x^2+1}$. c. $f(x) = \ln(x^2 + 2x + 10)$. d. $f(x) = \frac{e^{\frac{1}{2}x^2}}{x}$, $x \neq 0$.

1.6 Monotonicity of Functions

In the previous section we showed that in case a differentiable function has on an open set an optimum at $x = a$, a has to be a stationary point of the function. Therefore, to determine optima of a differentiable function, we look for stationary points of that function. However, as we already illustrated in Figures 1.13 and 1.14, in case we find a stationary point at $x = a$, it is unclear whether the function will have a maximum, minimum or neither of these at $x = a$. Therefore, additional criteria are required to determine whether a stationary point actually corresponds to an optimum. Before introducing these criteria, we first examine the conditions under which a function is increasing or decreasing.

DEFINITION 1.32 Let f be defined on a set S . We say that f :

- *increases* on the set S if for each $x, y \in S$ with $x < y$: $f(x) \leq f(y)$.
- *strictly increases* on the set S if for each $x, y \in S$ with $x < y$: $f(x) < f(y)$.
- *decreases* on the set S if for each $x, y \in S$ with $x < y$: $f(x) \geq f(y)$.
- *strictly decreases* on the set S if for each $x, y \in S$ with $x < y$: $f(x) > f(y)$.

A function which is either (strictly) increasing (decreasing) is called *monotone*.

Figure 1.15 illustrates the various notions of monotonicity.

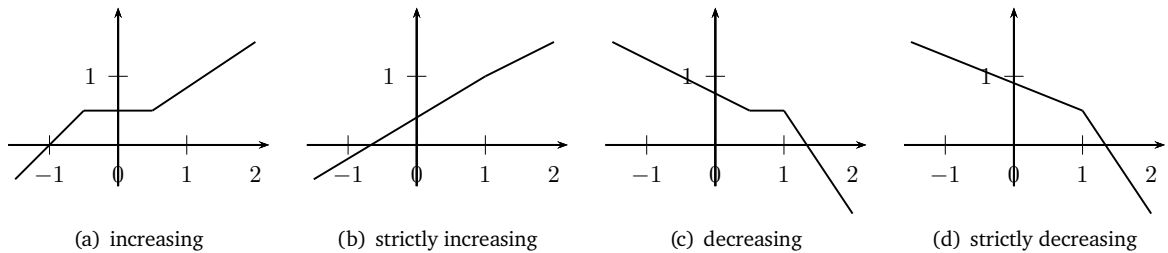


FIGURE 1.15 – Monotonicity of functions.

Notice that even in case a function is strictly decreasing everywhere this does not imply that the function values becomes arbitrarily negative. It may happen that the function values converge to some limiting value. Consider, e.g., the function $f(x) = \frac{1}{x}$ for $x > 0$. This function decreases everywhere and converges for large values of x to zero (though it never reaches this value), see Figure 1.16.

From Figure 1.15 we observe the next properties about the sign of the derivatives of the monotonic functions, which can be shown to hold in general if a function is differentiable (see also Exercise 1.37).

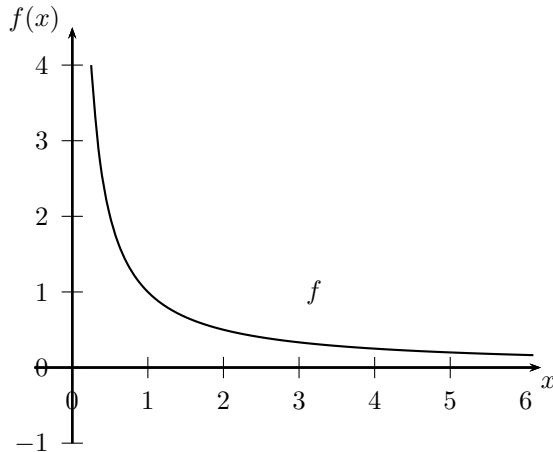


FIGURE 1.16 – Graph of $f(x) = \frac{1}{x}$ for $x > 0$.

THEOREM 1.33 Let f be differentiable on the open interval I . Then,

- f is increasing on $I \Leftrightarrow f'(x) \geq 0$.
- f is decreasing on $I \Leftrightarrow f'(x) \leq 0$.
- $f'(x) > 0 \Rightarrow f$ is strictly increasing.
- $f'(x) < 0 \Rightarrow f$ is strictly decreasing.

Figure 1.13 clearly illustrates that a function f has a maximum at $x = a$ if to the left of $x = a$ the function is increasing and to the right of $x = a$ the function is decreasing. We formulate this observation in Theorem 1.34. Figure 1.17 illustrates the theorem.

THEOREM 1.34 (sign scheme of the derivative)

Let f be differentiable on an open interval I and $a \in I$. If $x = a$ is a stationary point of f , then

- f has a maximum at $x = a$ if there exist numbers $a_l < a$ and $a_r > a$ such that
 1. $f'(x) \geq 0$ for $a_l \leq x \leq a$ and
 2. $f'(x) \leq 0$ for $a \leq x \leq a_r$. (See Figure 1.17, panel (a).)
- f has a minimum at $x = a$ if there exist numbers $a_l < a$ and $a_r > a$ such that
 1. $f'(x) \leq 0$ for $a_l \leq x \leq a$ and
 2. $f'(x) \geq 0$ for $a \leq x \leq a_r$. (See Figure 1.17, panel (b).)

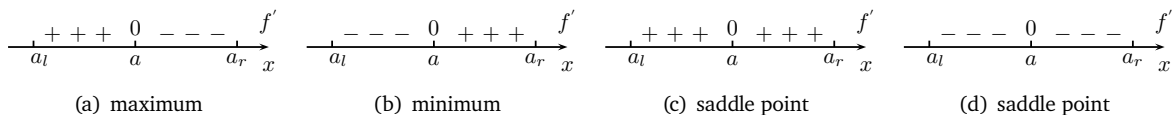


FIGURE 1.17 – Sign scheme of the derivative.

REMARK 1.35 Unfortunately there exist situations where it is impossible to draw a sign scheme of the derivative of the function f , even if this function is infinitely many times differentiable. This occurs if every open interval that contains the stationary point $x = a$ has additional stationary points, whereas the function is not constant. Section 1.8 provides examples of such functions.

EXERCISE 1.11 Determine the intervals where the next functions $f(x)$ are decreasing and increasing, respectively.

- a. $f(x) = x^3 - 3x$. b. $f(x) = e^{x^2}$. c. $f(x) = \frac{1}{x^2}, x \neq 0$. d. $f(x) = \frac{e^x}{x^2}, x \neq 0$.

EXERCISE 1.12 Determine the optima of the next functions, using Theorem 1.34.

- a. $f(x) = x^3 - 3x^2 + 3x + 1$. c. $f(x) = \begin{cases} -(x+1)^2, & x \leq -1; \\ 0, & x \in (-1, 0); \\ x^2, & x \geq 0. \end{cases}$
- b. $f(x) = x^7 - \frac{7}{5}x^5$.

EXERCISE 1.13 Determine the optima of the next functions $f(x)$. Indicate whether the optima are local or global optima.

- a. $f(x) = x^3 - 3x + 1$. c. $f(x) = xe^{-\frac{1}{2}x^2}, -2 \leq x \leq 2$.
- b. $f(x) = 3x^4 - 4x^3 + 5$. d. $f(x) = \frac{x}{1+x^2}$.

1.7 Convexity and Second-Order Condition for an Optimum of a Scalar Function

In Section 1.5, we argued that in order to identify the extremal points of a differentiable function f on an open interval, one should first determine the stationary points of f . That is, by computing the derivative of $f(x)$, equating it to zero, and solving for x , we obtain all candidates for the maxima and minima of $f(x)$. In the previous section we argued how, based on the sign scheme of the derivative, one can decide whether a stationary point will be a maximum, minimum or saddle point location.

In this section we analyze the behavior of functions near optimal points in some more detail. This will ultimately lead us to the formulation of the, so-called, second-order condition for an optimum. To that end reconsider the local maximum shown in Figure 1.18 again.

Roughly speaking, f has a local maximum at a since in this graph we are at the top of a "hill" at $x = a$. Similarly, f has a local minimum at b since in this graph we are at the lowest point of a "valley" at $x = b$. We shall next give a more accurate formalization of what characterizes a "hill" and "valley", respectively.

First, consider a "hill". Geometrically, a "hill" is characterized by the fact that every point on any line, connecting two points of the graph of the "hill", lies on or below the graph of the "hill". Figure 1.19 illustrates the situation.

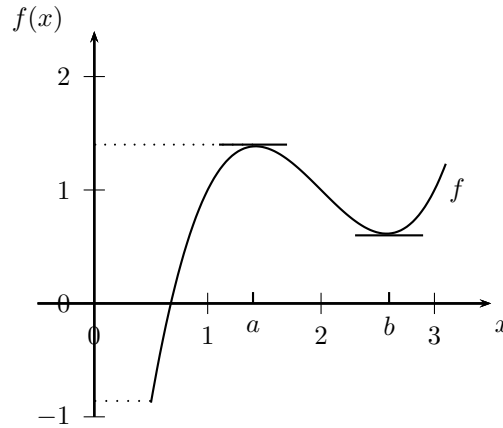


FIGURE 1.18 – Optima of a function.

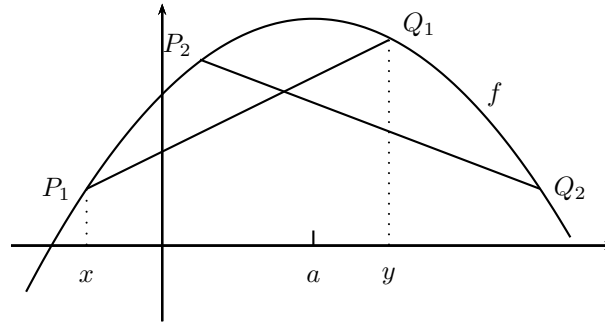


FIGURE 1.19 – Characterization of a maximum.

If we consider in this figure, e.g., the line P_1Q_1 , we see that every point on this line has the property that it lies on or beneath the graph of f . To be somewhat more specific: consider the interval $[x, y]$. We can parameterize this interval as: $\lambda x + (1 - \lambda)y$, $\lambda \in [0, 1]$. On this interval we have that at any point $s := \bar{\lambda}x + (1 - \bar{\lambda})y$, $\bar{\lambda} \in [0, 1]$, the value on the line P_1Q_1 is less or smaller than the value of f . Note that the line P_1Q_1 is described by the formula

$$m(s) = f(x) + \frac{f(y) - f(x)}{y - x}(s - x).$$

So, we get:

$$\begin{aligned} f(s) &\geq m(s), \text{ or,} \\ f(\bar{\lambda}x + (1 - \bar{\lambda})y) &\geq m(\bar{\lambda}x + (1 - \bar{\lambda})y), \text{ or,} \\ f(\bar{\lambda}x + (1 - \bar{\lambda})y) &\geq f(x) + \frac{f(y) - f(x)}{y - x}(\bar{\lambda}x + (1 - \bar{\lambda})y - x), \text{ or,} \\ f(\bar{\lambda}x + (1 - \bar{\lambda})y) &\geq \bar{\lambda}f(x) + (1 - \bar{\lambda})f(y). \end{aligned}$$

Functions that have this geometric property are called concave. In case the opposite property holds, i.e., every point on a line connecting two points on the graph of a function f is located above this graph, a function is called convex. Similarly as above one can show that f is convex if and only if the next inequality holds for all x, y and $\bar{\lambda} \in [0, 1]$:

$$f(\bar{\lambda}x + (1 - \bar{\lambda})y) \leq \bar{\lambda}f(x) + (1 - \bar{\lambda})f(y).$$

DEFINITION 1.36 Let f be defined on an interval I . Then, f is called

- *convex* on I if it is true that, for each $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

whenever $x \in I$ and $y \in I$.

- *concave* on I if it is true that, for each $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y),$$

whenever $x \in I$ and $y \in I$.

If a strict inequality holds for all $\lambda \in (0, 1)$, and $x \neq y$, we say that f is *strictly convex* (concave, respectively).

Figure 1.20 illustrates the above introduced concepts.

From the definitions it is easily verified that the next relationship between convex and concave functions holds.

PROPOSITION 1.37 Let f be defined on an interval I . Then, f is concave on I if and only if $-f$ is convex on I .

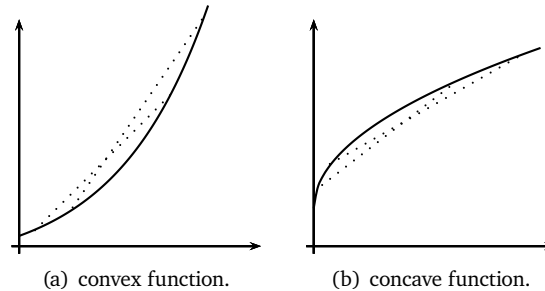


FIGURE 1.20 – *Convex/Concave functions.*

It can be shown that every convex (concave) function is always continuous. That a convex function is not always differentiable demonstrates the example $f(x) = |x|$ (see Figure 1.21).

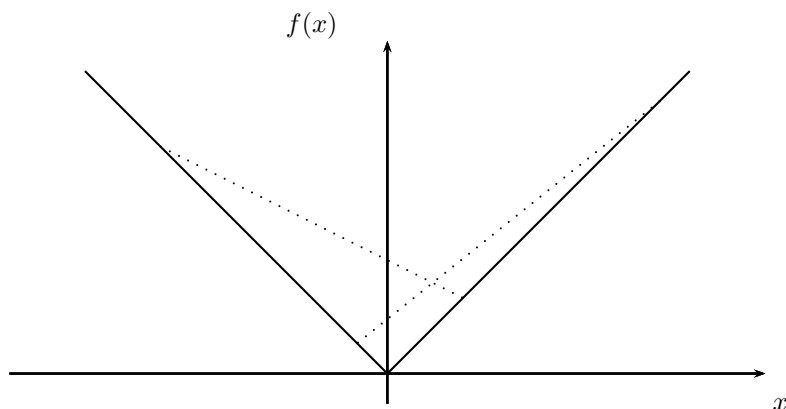


FIGURE 1.21 – Non-differentiable convex function $f(x) = |x|$.

Figure 1.20, panel (a), illustrates moreover, that a differentiable convex function f has the property that the slope of the tangent line increases if x increases. We will exploit this observation in a moment to derive a second-order characterization for a convex function.

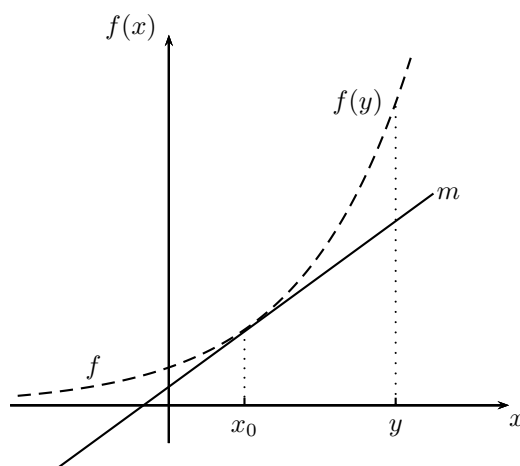


FIGURE 1.22 – Characterization convex differentiable function f .

First, however, another observation. To that end consider Figure 1.22. In this graph we plotted at an arbitrary point the tangent line, $m(x)$, to the graph of the convex function f . What we observe is that the graph of f is located completely above this tangent line $m(x)$. It can be shown (see Exercise 1.38) that this property characterizes convex functions that are differentiable. An important practical consequence is that in case we know for a convex function f just its value and its derivative at some point x_0 , we can use this information to find easily an underestimate for the value of f at any point x of its domain. That is, we can estimate this value by the value attained on the tangent line at y : $f(y) \geq m(y) = f(x_0) + f'(x_0)(y - x_0)$. In case one has collected this information about value and its derivative for this convex function for a number of different points, one can easily get an impression

from all these corresponding tangent lines how the complete graph of f will look like. Theorem 1.38 formalizes this observation

THEOREM 1.38 (*First-Order Characterization convex/concave function*) *Let I be an open interval and suppose f is differentiable on I . Then, f is*

- *convex on I if and only if $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in I$;*
- *concave on I if and only if $f(y) \leq f(x) + f'(x)(y - x)$ for all $x, y \in I$.*

As noted earlier, another observation—illustrated for example by the graph in Figure 1.20, panel (a)—is that for a differentiable convex function f , the slope of the tangent line increases as x increases. Or, stated differently, the derivative of f increases. This implies that the derivative of f' , the so-called second-order derivative of the function f (notation: f''), is nonnegative. In practice, this property provides a very useful characterization of a convex function when f is twice differentiable on an open interval I . Therefore, we give a precise statement in Theorem 1.39 below.

THEOREM 1.39 (*Second-Order Characterization convex/concave function*) *Let I be an open interval and suppose f is twice differentiable on I . Then, f is*

- *convex on I if and only if $f''(x) \geq 0$ for all $x \in I$;*
- *concave on I if and only if $f''(x) \leq 0$ for all $x \in I$.*

A direct consequence of the above characterization is that if two functions that are twice differentiable are convex (concave), their sum will be convex (concave) too. This, since $(f + g)''(x) = f''(x) + g''(x)$.

EXERCISE 1.14 Indicate the intervals where the next functions are convex and concave, respectively.

- a. $f(x) = e^x$. b. $f(x) = \ln(x)$, $x > 0$. c. $f(x) = e^{-\frac{1}{2}x^2}$. d. $f(x) = e^x - \ln(x)$, $x > 0$.

Note that a line is both convex and concave. In fact, this is the only function that can have both properties simultaneously.

Next, we return to the second item of this section, i.e., to characterize maxima and minima of a function. Clearly, a function will have a maximum (minimum) at a stationary point a if the function is concave (convex) on an open interval containing a . So, using our previous characterization of concavity of functions, we get the next result.

THEOREM 1.40 *Let f be twice differentiable on an open interval I and $a \in I$. Assume $f'(a) = 0$. Then, f has a local*

- *maximum at a if $f''(x) \leq 0$ on an open interval containing a ;*
- *minimum at a if $f''(x) \geq 0$ on an open interval containing a .*

REMARK 1.41 Unfortunately, the converse of Theorem 1.40 does not hold in general. That is, the intuition that if a function has, for instance, a local minimum at $x = a$ the function will be locally convex around $x = a$ is not correct. The reason for this is along the lines of the observation we made in Remark 1.35. There exist nonconstant functions f that have the property that in every open interval of its minimum location $x = a$, there are additional minima and maxima. In Section 1.8 we will provide an example of such a function. Moreover, a sufficient condition will be given from which one can conclude that around a minimum location the function will be locally convex.

Since, especially from a numerical perspective, verifying the sign of a function over an interval is cumbersome, it would be convenient to have a criterion that determines the nature of a stationary point by examining only the second-order derivative of the function at that point. Such a result immediately follows from the previous theorem. For, if a function is twice differentiable and this second-order derivative function is continuous, it follows directly that if, e.g., $f''(a) < 0$, obviously $f''(x) < 0$ on an open interval containing a too. So, by Theorem 1.40, we immediately conclude that in this case the function attains a maximum at $x = a$. In literature this result is known as the *second-order criterion*. Without going into details, we note that the above assumption that the second-order derivative of f is continuous can be dropped.

THEOREM 1.42 (*second-order criterion*) Let f be twice differentiable on an open interval I containing a . Assume f has a stationary point at $x = a$. Then $f(a)$ will be

- a local maximum if $f''(a) < 0$;
- a local minimum if $f''(a) > 0$.

The next example shows that no unequivocal conclusion can be drawn about the optimality of the stationary point a in case $f''(a) = 0$. In such a case the function can either attain a maximum, a minimum or a saddle point at $x = a$.

EXAMPLE 1.43

1. Consider $f(x) = x^4$. Clearly f has a stationary point at $x = 0$, $f'(0) = 0$ and f has a global minimum at $x = 0$ since $f(x) \geq 0$.
2. Consider $f(x) = -x^4$. Clearly f has a stationary point at $x = 0$, $f'(0) = 0$ and f has a global maximum at $x = 0$ since $f(x) \leq 0$.
3. Consider $f(x) = x^3$. Clearly f has a stationary point at $x = 0$, $f'(0) = 0$ and f has a saddle point at $x = 0$.

REMARK 1.44 In case all derivatives of f up to some order $n - 1$ are zero at $x = a$, i.e. $f^{(k)}(a) = 0$, $k = 1, \dots, n - 1$, but $f^{(n)}(a) \neq 0$ and, moreover, $f^{(n)}(x)$ is continuous we have next result (see Exercise 1.48):

1. If n is even and $f^{(n)}(a) > 0$, f has a local minimum at $x = a$.
2. If n is even and $f^{(n)}(a) < 0$, f has a local maximum at $x = a$.
3. If n is odd and $f^{(n)}(a) \neq 0$, f has a saddle point at $x = a$.

EXERCISE 1.15 Verify the correctness of the result stated in Remark 1.44 by checking the advertized conditions for the examples considered in Example 1.43.

EXERCISE 1.16 Consider the function $f(x) = 1 + (x^2 - 1)^3$. Determine all extremal points of f and indicate whether they are local or global optima.

EXERCISE 1.17 Show that the function $f(x) = x^8 - 4x^2$ has a local maximum at $x = 0$. Is this also a global maximum? Determine all extremal points of f on $[-1.1, 1.1]$.

1.8 Sexi Functions

In this section, we examine several examples that illustrate why certain properties of functions - properties one might naturally expect to hold - do not in fact apply. The examples are constructed by combining the periodic function $s(x) = \sin x$, the exponential function $ex(x) = e^{-x^2}$, and the function $i(x) = \frac{1}{x}$. A combination of such functions is called a Sexi function.

Below, we briefly discuss some important properties of these three constituent functions.

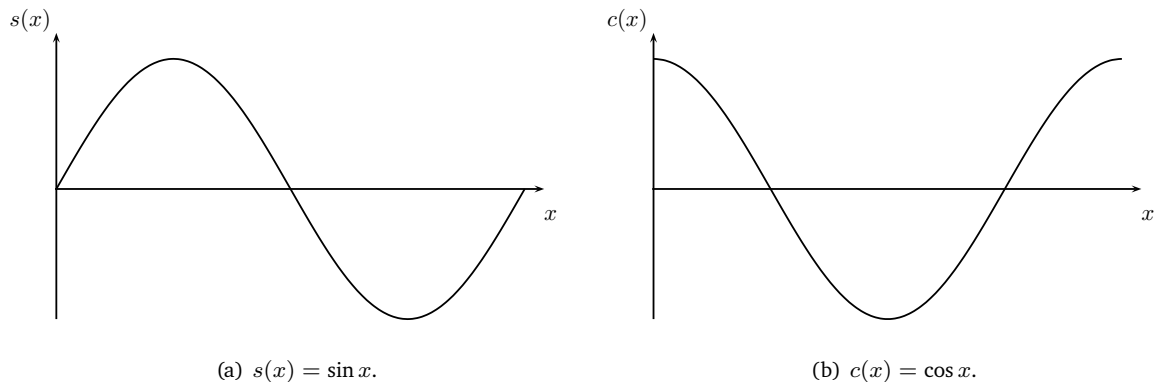


FIGURE 1.23 – Goniometric functions.

EXAMPLE 1.45 First we recall some elementary properties of goniometric functions $s(x) = \sin x$ and $c(x) = \cos x$, respectively. We plotted them in Figure 1.23.

Both functions repeat with a periodicity of 2π and their range is $[-1, 1]$. $s(x)$ has a maximum at $\frac{1}{2}\pi$ and a minimum at $\frac{3}{2}\pi$. Furthermore, $s(x) = 0$ at $x = 0$ and $x = \pi$. $c(x)$ attains its maximum at $x = 0$; its minimum at $x = \pi$; and $c(\frac{1}{2}\pi) = c(\frac{3}{2}\pi) = 0$. Both $s(x)$ and $c(x)$ are differentiable. It can be shown that

$$s'(x) = \cos x \text{ and } c'(x) = -\sin x.$$

Next, consider the function $ex(x) = e^{-x^2}$.

EXAMPLE 1.46 Consider $ex(x) = e^{-x^2}$. Observe that this function is symmetric with respect to the y-axis; that is, $ex(x) = ex(-x)$. Consequently, the graph left to the y-axis is obtained by reflection through the y-axis of the graph to the right of the y-axis. The derivative of this function is

$$ex'(x) = -2xe^{-x^2}.$$

So, $ex(x)$ has one stationary point at $x = 0$. From the sign scheme of ex' it follows that $ex(0) = 1$ is a global maximum. Moreover, $ex(x)$ converges to zero if either x goes to infinity or minus infinity. We plotted the graph of $ex(x)$ in Figure 1.24.

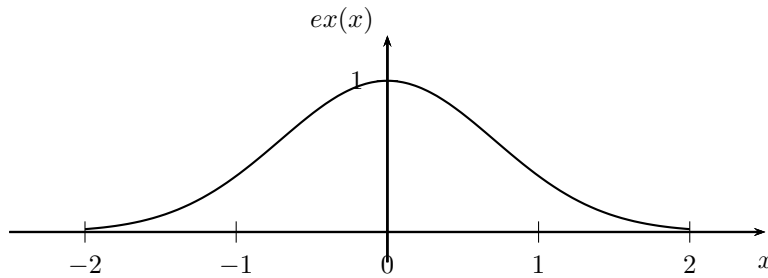


FIGURE 1.24 – Function $ex(x) = e^{-x^2}$.

We already plotted the graph of the function $i(x)$ for $x > 0$ in Figure 1.16. As $i(-x) = -\frac{1}{x} = -i(x)$, it follows directly that the graph left to the y-axis of $i(x)$ is obtained by reflection through the line $y = -x$ of Figure 1.16. We plotted the complete graph of $i(x)$ in Figure 1.25. Notice that $i(x)$ is not defined at $x = 0$ and that $i(x)$ is infinitely many times differentiable for all $x \neq 0$. Furthermore, $i(i(x)) = id(x)$. That is, the inverse function of $i(x)$ is $i(x)$.

To create some intuition for the upcoming examples we first consider in Example 1.47, below, the graph of the product of the functions $ex(x)$ and $s(x)$, i.e., of function $f(x) = e^{-x^2} \sin x$.

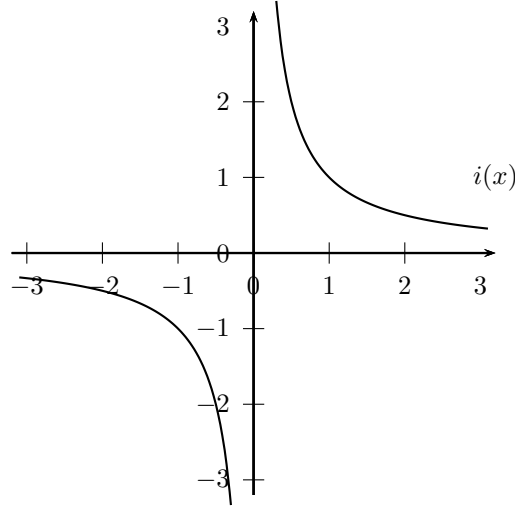


FIGURE 1.25 – Graph of $i(x) = \frac{1}{x}$ for $x \neq 0$.

EXAMPLE 1.47 Consider $f(x) = e^{-x^2} \sin x$. Geometrically, the graph of this function coincides with the graph of $s(x) = \sin x$. The only difference between $f(x)$ and $s(x)$ is that the value of $f(x)$ is obtained by multiplying the value of $s(x)$ by the number e^{-x^2} which, see the graph of Figure 1.24, converges exponentially to zero if either x goes to infinity or minus infinity. In particular this implies that the zeros and extremal points of $f(x)$ coincide with those of $\sin x$. Therefore, the maximum value of $f(x)$ at $x^* = (\frac{1}{2} + 2k)\pi$, $k = \dots, -2, -1, 0, 1, 2, \dots$ is $e^{-x^{*2}}$ and the minimum value of $f(x)$ at $x_* = (\frac{3}{2} + 2k)\pi$, $k = \dots, -2, -1, 0, 1, 2, \dots$ is $-e^{-x_*^2}$. The range of $s(x)$ is $[-1, 1]$; the extremal points of $f(x)$ lie between $-1 * e^{-x(x)}$ and $1 * e^{-x(x)}$. We plotted the graph of $f(x)$ in Figure 1.26.

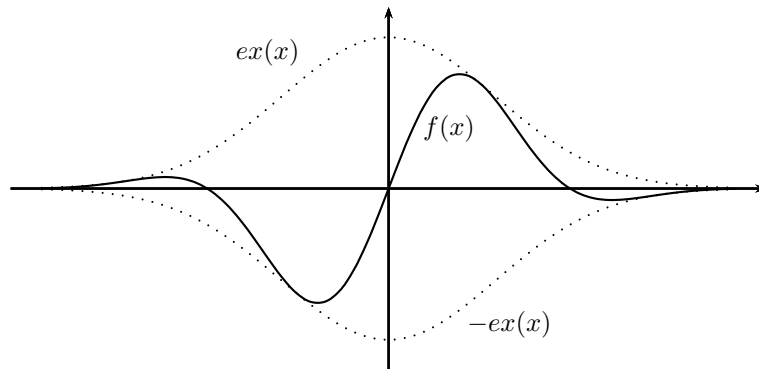


FIGURE 1.26 – Function $f(x) = e^{-x^2} \sin x$.

Our first, perhaps counterintuitive, Example 1.48 below demonstrates that a function which is infinitely differentiable can possess infinitely many extremal points on a bounded interval. In particular, this makes it impossible to construct a sign scheme for its derivative.

EXAMPLE 1.48 Consider the composite function $f(x) = (s \circ i)(x) = \sin \frac{1}{x}$, $x \neq 0$. As both $s(x)$ and $i(x)$ are infinitely many times differentiable, by Theorem 1.24, $f(x)$ has this property too. Clearly the values of $f(x)$ fluctuate between -1 and 1 . Furthermore, the closer x approaches zero, the more frequent $f(x)$ fluctuates between these extremal values. In fact, for all integers $k \neq 0$, $f(\bar{x}) = 0$ at $\bar{x} = \frac{1}{k\pi}$; $f(x^*) = 1$ at $x^* = \frac{1}{(\frac{\pi}{2} + 2k)\pi}$; and $f(x_*) = -1$ at $x_* = \frac{1}{(\frac{3\pi}{2} + 2k)\pi}$. So, for any number $a > 0$, f has an infinite number of extremal points on the interval $(0, a)$. In fact at all these extremal points f attains a global extremal value.

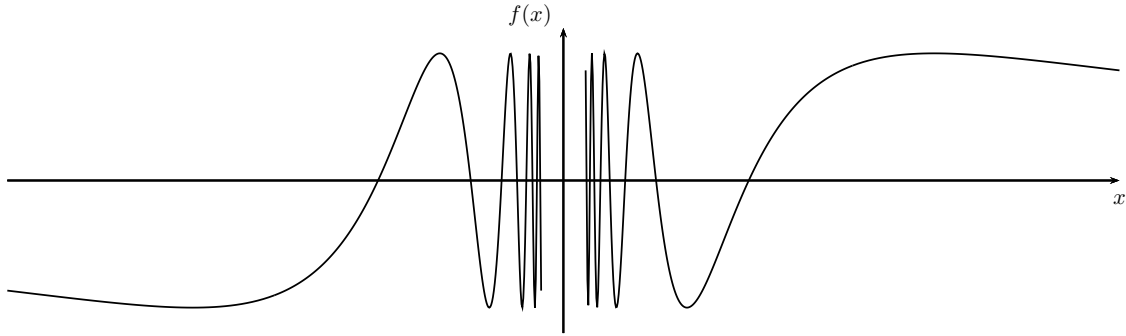


FIGURE 1.27 – Function $f(x) = \sin \frac{1}{x}$, $x \notin (-0.04, 0.04)$.

Notice that for $x > \frac{2}{\pi}$, $f(x) > 0$, and converges monotonically to zero if x goes to ∞ . Similarly for $x < -\frac{2}{\pi}$, $f(x) < 0$, and converges monotonically to zero too if x goes to $-\infty$. Figure 1.27 illustrates the graph.

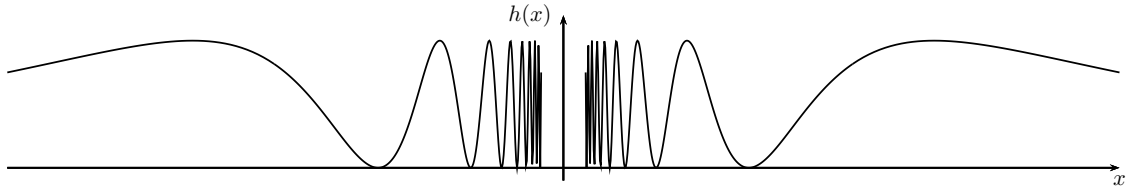


FIGURE 1.28 – Function $h(x) = \sin^2 \frac{1}{x}$, $x \notin (-0.04, 0.04)$.

From this function $f(x)$ we can easily construct a nonnegative function $h(x)$ that is an infinite number of times differentiable and which attains in any interval $(0, a)$, at an infinite many number of points, its global minimum value 0. Consider, e.g., $h(x) = f^2(x) = \sin^2 \frac{1}{x}$. Clearly, the graph of this function is obtained by squaring all numbers in Figure 1.27. We plotted a part of the resulting graph in Figure 1.28. Notice that $h(x)$ is symmetric w.r.t. the y-axis, i.e. $h(x) = h(-x)$, since $s(y) = \sin y$ has the property that $s(-y) = -s(y)$.

EXAMPLE 1.49 As a final preparatory example we consider

$$g(x) = (ex \circ i)(x) = e^{-\frac{1}{x^2}}, \quad x \neq 0, \quad \text{and } g(0) = 0.$$

Notice that $g(x)$ is symmetric with respect to the y-axis as, $g(x) = g(-x)$. So the graph left to the y-axis is obtained by reflection through the y-axis of the graph to the right of the y-axis.

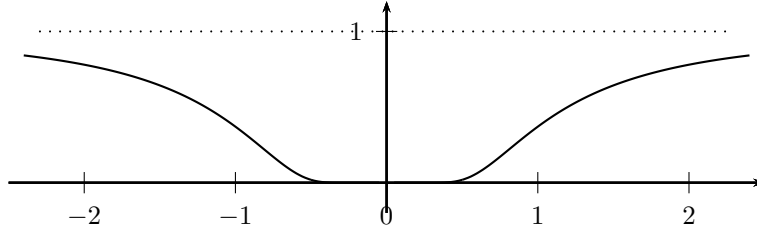


FIGURE 1.29 – Function $g(x) = e^{-\frac{1}{x^2}}$.

Basically, the graph of g on $(0, \infty)$ is obtained by copying the graph of ex from $(\infty, 0)$. We plotted the graph of $g(x)$ in Figure 1.29. Notice that, by Theorem 1.24, $g(x)$ is infinitely many times differentiable for all $x \neq 0$. Using formal analysis, which is beyond the scope of this book, it can be shown that g is also infinitely many times differentiable at $x = 0$.

Next Example 1.50 shows that a function that has a minimum is not necessarily convex in a neighborhood of that minimum.

EXAMPLE 1.50 Consider

$$f(x) = e^{-\frac{1}{x^2}} \sin^2 \frac{1}{x}, \quad x \neq 0, \quad \text{and} \quad f(0) = 0.$$

Notice that $f(x)$ is symmetric with respect to the y-axis as, $f(x) = f(-x)$. So the graph left to the y-axis is obtained by reflection through the y-axis of the graph to the right of the y-axis. Moreover, by Theorem 1.24, $f(x)$ is infinitely many times differentiable for all $x \neq 0$ and, formal analysis (which again is beyond the scope of this book) shows that, $f(x)$ has this property at $x = 0$ too.

Similar to Example 1.47, the graph of $f(x)$ is obtained as the graph of $h(x) = \sin^2 \frac{1}{x}$. The only difference between $f(x)$ and $h(x)$ is that the value of $f(x)$ is obtained by multiplying the value of $h(x)$ by the number $e^{-\frac{1}{x^2}}$ which, see the graph of Figure 1.29, converges exponentially to zero if x goes to zero. Notice $f(x)$ is nonnegative everywhere. In particular this implies that the points where $f(x)$ attains a global minimum coincide with those where $h(x)$ attains a value of zero. That is, for $x_* = \frac{1}{k\pi}$ (where k is any integer) and $x_* = 0$, f attains its global minimum $f(x_*) = 0$. Figure 1.30 illustrates the graph. Unfortunately, due to the very fast convergence of $e^{-\frac{1}{x^2}}$ towards zero, for small values of x the fluctuations as demonstrated in Figure 1.28 are not visible.

Though this function has a global minimum at $x = 0$, $f(x)$ is not locally convex at $x = 0$. Whatever interval $(-a, b)$ we choose around $x = 0$, there is a point $x_* = \frac{1}{k\pi} \in (-a, b)$ where $f(x_*) = 0$. As $f(x) \geq 0$, it follows that the line connecting x_* and $x = 0$ is located below the graph of f . So, f does not satisfy the definition of convexity.

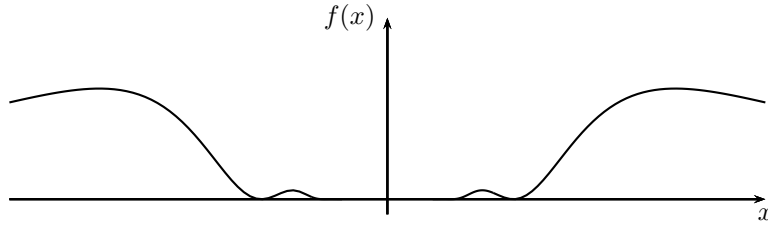


FIGURE 1.30 – Function $f(x) = e^{-\frac{1}{x^2}} \sin^2 \frac{1}{x}$.

Next, we provide an example of a function $f(x)$ that has a saddle point at $x = 0$, but for which we cannot find an interval $(0, a)$, $a > 0$, where this function either always assumes positive values or only negative values.

EXAMPLE 1.51 Consider

$$f(x) = e^{-\frac{1}{x^2}} \sin \frac{1}{x}, \quad x \neq 0, \quad \text{and} \quad f(0) = 0.$$

Notice that $f(x)$ is asymmetric with respect to the y-axis as, $f(x) = -f(-x)$. So the graph left to the y-axis is obtained by reflection through the line $y = -x$ of the graph to the right of the y-axis. By Theorem 1.24, $f(x)$ is infinitely many times differentiable for all $x \neq 0$ and, formal analysis (which again is beyond the scope of this book) shows that, $f(x)$ has this property at $x = 0$ too. Moreover, it can be shown that $f'(0) = 0$.

Similar to Example 1.47, the graph of $f(x)$ is obtained as the graph of $h(x) = \sin \frac{1}{x}$. The only difference between $f(x)$ and $h(x)$ is that the value of $f(x)$ is obtained by multiplying the value of $h(x)$ by the number $e^{-\frac{1}{x^2}}$ which, see the graph of Figure 1.29, converges exponentially to zero if x goes to zero. In particular this implies that the extremal points of $f(x)$ coincide with those of $h(x)$. Figure 1.31 visualizes the graph of $f(x)$. Like in Figure 1.30, it fails to provide a good picture of what is going on for small values of x , since in that case the number $e^{-\frac{1}{x^2}}$ is very small. This makes it difficult to visualize the fluctuating behaviour that occurs near $x = 0$ as demonstrated in Figure 1.27.

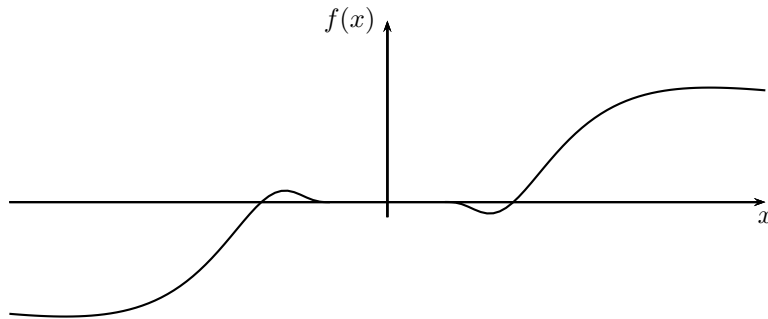


FIGURE 1.31 – Function $f(x) = e^{-\frac{1}{x^2}} \sin \frac{1}{x}$.

In particular, every interval $(-a, b)$ containing $x = 0$, contains a point where h attains a maximum value of 1 and a point where h attains a minimum value of -1 . Therefore, f attains neither a maximum nor a minimum at $x = 0$. So, by definition, $x = 0$ is a saddle point. However, notice that the geometric picture associated with this name in this case breaks down. That is, neither to the left of $x = 0$ there is an interval $(-b, 0)$ such that $f(x) \geq 0$ everywhere in this interval, or to the right of $x = 0$ there is an interval $(0, a)$ where $f(x) \geq 0$ everywhere in this interval.

The previous two examples demonstrated that counterintuitive behaviour may occur around a stationary point in which neighborhood infinite many additional stationary points occur. A more close inspection of the stationary point $x = 0$ in both previous examples shows that this happens only in rare cases. In the examples we observe that in case we differentiate the function $f(x)$ k times, whatever k is, this derivative will be zero at $x = 0$. In case this phenomenon does not occur at a stationary point where an extremal point occurs, it can be shown that the function will be either locally convex or concave (see Exercise 1.49). Theorem 1.52, below, summarizes this result.

THEOREM 1.52 Assume $f(x)$ has a minimum at $x = a$ and $f^{(k)}(a) = 0$, $k = 1, \dots, n-1$, but $f^{(n)}(a) \neq 0$. Assume, moreover, that $f^{(n)}(x)$ is continuous in an open interval I_1 containing $x = a$. Then, $f(x)$ is locally convex on an open interval I containing $x = a$.

EXAMPLE 1.53 Consider $f(x) = x^4$. As $f(x) \geq 0$ and $f(0) = 0$ it is obvious that f has a minimum at $x = 0$. Notice that $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f^{(3)}(x) = 24x$, $f^{(4)}(x) = 24$ and $f^{(k)}(x) = 0$ for $k > 5$. So, $f^{(4)}(0) \neq 0$ and it follows by Theorem 1.52 that $f(x)$ is locally convex around $x = 0$.

EXERCISE 1.18 Consider the function $f(x) = x^2 \sin \frac{1}{x}$, $x \neq 0$. Let $f(0) = 0$.

- Use the definition of differentiability to show that $f(x)$ is differentiable at $x = 0$.
- Show that $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, $x \neq 0$.
- Show that $f'(x)$ is not continuous at $x = 0$.

Mixed Exercises

EXERCISE 1.19 Determine for the functions f below all stationary points. Indicate whether f attains at these points a local or global minimum/maximum.

- | | | |
|--|----------------------------------|--|
| a. $f(x) = x^2 - 6x + 8$. | d. $f(x) = x^5 + 1$. | g. $f(x) = \frac{e^x}{1+x^2}$. |
| b. $f(x) = x^4 - 2x^2 + 2$. | e. $f(x) = \frac{x}{e^x}$. | h. $f(x) = \frac{2x^2+4}{\frac{1}{4}x^4+\frac{1}{2}x^2+1}$. |
| c. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x - 1$. | f. $f(x) = \frac{4x+3}{x^2+1}$. | |

EXERCISE 1.20 Determine for the functions g below all stationary points. Indicate whether g attains at these points a local or global minimum/maximum⁴.

a. $g(x) = \log(1 + x^2)$. b. $g(x) = e^{x^2+2x}$. c. $g(x) = \sqrt{\frac{1}{4}x^4 + x^2 + 1}$. d. $g(x) = xe^{-\frac{1}{2}x^2}$.

EXERCISE 1.21 Determine for the functions h below all stationary points. Indicate whether h attains at these points a local or global minimum/maximum.

a. $h(x) = (e^{x^2} + 1)^2$. b. $h(x) = \frac{x\sqrt{x}}{e^{-\frac{1}{2}x+1}}$, $x \geq 0$. c. $h(x) = (x^2 + 1)^{\frac{1}{3}}(x^2 + 2)^{\frac{2}{3}}$.
d. $h(x) = \ln(\frac{1}{1+x^2}) + \ln(\sqrt{1+x^2})$.

EXERCISE 1.22 Determine for the functions f below all extremal points. Indicate whether f attains at these points a local or global minimum/maximum.

a. $f(x) = x^2 - 4x + 3$, $x \in [0, 5]$. e. $f(x) = \frac{x^2}{1-x}$, $x \in [\frac{3}{2}, 5]$.
b. $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$, $x \in [0, 4]$. f. $f(x) = \frac{x^2}{1+x}$, $x \in [-4, -1) \cup (-1, 4]$.
c. $f(x) = \frac{1}{3}x^3 - x + 1$, $x \in [-2, 2]$. g. $f(x) = \frac{1}{1+x^2}$, $x \in [-1, \infty)$.
d. $f(x) = \ln(1+x)$, $x \in [0, 10]$. h. $f(x) = \frac{x}{1+x^2}$, $x \in [-\frac{1}{2}, \infty)$.

EXERCISE 1.23 Solve the next equations.

a. $2 * 2^x + 4 * 2^x = 24$. d. $\ln(e^{x^2}) + e^{\ln(x)} = 2$.
b. $2^{2x} + 2^{x+2} = 5$. e. $\ln(5x - 7) - \ln(x - 1) = \ln(x + 1)$, $x > \frac{7}{5}$.
c. $\frac{1}{4} * 2^{2x} + 4^x + \frac{3}{4^{1-x}} = 32$. f. $\ln(x^2 - 2x + 1) = \frac{2}{\ln(x-1)}$, $x > 1$.

EXERCISE 1.24 Determine the boundary and accumulation points of next sets. Motivate whether these sets are bounded/open/closed or compact, respectively

a. $(-1, 2] \cup (3, 4)$. b. $[-1, 2) \cup (2, 4]$. c. $[-1, 2] \cup [3, 4]$. d. $(-1, 2) \cup (2, \infty)$.

EXERCISE 1.25 Determine the largest domain for next functions g where they are continuous, respectively differentiable. Motivate your answer.

a. $g(x) = x + \ln(1 + x^2)$. b. $g(x) = |x + 1| + e^{x-x^2}$. c. $g(x) = \frac{\sqrt{x-1}}{x}$.
d. $g(x) = \frac{\ln(1+x)}{x}$. e. $g(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \neq 1; \\ 0, & x = 1. \end{cases}$ f. $g(x) = \begin{cases} x^2 + 1, & x \leq 0; \\ |x + 1|, & x > 0. \end{cases}$

EXERCISE 1.26 Determine the intervals where the next functions h are decreasing and increasing, respectively.

a. $h(x) = x^2 - 4x + 4$. c. $h(x) = x^4 - \frac{8}{3}x^3 + 2x^2 + 1$. e. $h(x) = \ln(1+x)$, $x > 0$.
b. $h(x) = -x^3 + 3x^2 - 3x + 1$. d. $h(x) = e^{-x^2+2x}$.

EXERCISE 1.27 Determine for the functions h in exercise 1.26 the intervals where they are convex and concave, respectively.

⁴ For item d note: $g(x) \rightarrow 0$ if $x \rightarrow \pm\infty$

Advanced, Deepening and Broadening Exercises

EXERCISE 1.28 Consider the intervals below in $\mathbb{R}$. Show that next statements are true.

- The intervals (a, b) , (a, ∞) and $(-\infty, b)$ are open.
- The intervals $[a, b]$, $[a, \infty)$ and $(-\infty, b]$ are closed.
- The intervals $[a, b)$ and $(a, b]$ are neither open or closed.
- $\mathbb{R}$, is as well open as closed.

EXERCISE 1.29 Which of the next statements are true. Motivate your answer.

- Every differentiable function is continuous.
- There exist continuous functions which are not differentiable.
- Every continuous function attains both a global maximum and a global minimum on $[0, 2]$.
- There exist continuous functions that attain both a global maximum and a global minimum on $(0, 2)$.
- A bounded set which contains all its boundary points is compact.
- Every strictly increasing function is convex.
- Every boundary point is an accumulation point.

EXERCISE 1.30 (*uniqueness inverse function*)

Assume $f : D \rightarrow R$ is invertible. Show, using the definition, that the inverse function is uniquely determined. Hint: assume f has two inverse functions. Show they have to coincide.

EXERCISE 1.31 The polynomial $p(x)$ of degree 3 has zeros at $x = 1$ and $x = 2$. Furthermore, $p(3) = 4$. Determine $p(x)$. Hint: use the property that if a polynomial $p_n(x)$ of degree n has a zero at $x = a$, $p_n(x) = (x - a)p_{n-1}(x)$, where p_{n-1} is a polynomial of degree $n - 1$.

EXERCISE 1.32 (*Limits and continuity*)

The intuition that a sequence $\{s_n\}_{n=1}^{\infty}$ approaches the number s (or: the limit of the sequence $\{s_n\}_{n=1}^{\infty}$ is s) is formal defined as: for every $\epsilon > 0$ (however small that may be) we can find a number N_ϵ such that, for all $n > N_\epsilon$ the distance between s_n and s , that is $|s_n - s|$, is smaller than this ϵ .

Use this definition to show that

- $f(x) = 1$ is continuous on $\mathbb{R}$.
- $f(x) = x$ is continuous on $\mathbb{R}$.
- $f(x)$ is discontinuous at $x = 1$ if $f(x) = 2$ for $x \in [0, 1)$ and $f(x) = 3$ for $x \in [1, 2)$.

EXERCISE 1.33 (*Differentiability implies continuity*)

Formal a function f is called differentiable at x_0 if the difference quotient at this point approaches some number if the change in the function argument, Δx , approaches zero. Show that if f is differentiable at x_0 , this implies that $f(x_0 + \Delta x)$ approaches $f(x_0)$ if Δx approaches zero. Or, stated differently, f is continuous at x_0 .

EXERCISE 1.34 (*First-order condition optimum*)

Show, using the definition of differentiability (see Exercise 1.33), that if f is differentiable on an open interval I , and f attains a maximum at $x_0 \in I$, $f'(x_0) = 0$. Hint, show this by contradiction. That is, assume either $f'(x_0) > 0$ or $f'(x_0) < 0$ and show that under those assumptions f cannot attain a maximum at x_0 .

EXERCISE 1.35 (*Rolle's Theorem*)

Consider the continuous function f on the compact interval $[a, b]$. Assume $f(x)$ is differentiable on the open interval (a, b) and $f(a) = f(b)$. Below you are asked to show that there exists a point $c \in (a, b)$ for which $f'(c) = 0$.

- Motivate why f attains both a maximum M and a minimum m on $[a, b]$.
- Assume $f(a) = M$ and $f(b) = m$. Show that there exists a point $c \in (a, b)$ for which $f'(c) = 0$.
- Assume either M or m is not attained at an endpoint of the interval. Show that there exists a point $c \in (a, b)$ for which $f'(c) = 0$.
- Assume $h(0) = h(1) = 0$ and $h(x)$ is differentiable on $(0, 1)$. Consider the function $f(x) = e^{-x^3 - x^2 - x}h(x)$ on $[0, 1]$. Motivate, using Rolle's Theorem, why there exists a point $c \in (0, 1)$ such that $f'(c) = 0$.

EXERCISE 1.36 (*Mean Value Theorem*)

Consider the continuous function f on the compact interval $[a, b]$. Assume $f(x)$ is differentiable on the open interval (a, b) . Below, in item c., you are asked to show that there exists a point $c \in (a, b)$ for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

- Consider $g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$. Show that $g(a) = g(b)$.
- Use Rolle's Theorem (Exercise 1.35) to conclude that there exists a point $c \in (a, b)$ for which $g'(c) = 0$.
- Show, using item b., that there exists a point $c \in (a, b)$ for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- Assume $f(x)$ is continuous on $[0, 1]$, differentiable on $(0, 1)$ and $f'(x) \in [-2, 2]$. Show that $-2 + f(0) \leq f(1) \leq 2 + f(0)$.

EXERCISE 1.37 (*Monotonicity and derivative*)

Assume f is differentiable on an open interval I . Show, using the definition of differentiability (see Exercise 1.33), that f is increasing on I if and only if $f'(x) \geq 0$. Hint: use the mean value theorem (Exercise 1.36) to estimate the difference quotient.

EXERCISE 1.38 (*First-order characterization convex function*)

Assume f is differentiable on an open interval I . Below, in items c. and e., respectively, you are asked to show that f is convex on I if and only if $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in I$.

a. Consider x_0 and $y = x_0 + \Delta x \in I$. Use the definition of convexity to conclude that

$$(1 - \lambda)f(x_0 + \Delta x) \geq f(x_0 + (1 - \lambda)\Delta x_0) - f(x_0) + (1 - \lambda)f(x_0), \text{ for each } \lambda \in [0, 1].$$

b. Let $t := 1 - \lambda$. Show that we obtain the next inequality from item a.:

$$f(x_0 + \Delta x) \geq f(x_0) + \frac{f(x_0 + t\Delta x_0) - f(x_0)}{t\Delta x_0} \Delta x_0.$$

c. Use the definition of differentiability (see Exercise 1.33) to show from item b., that if f is convex, $f(y) \geq f(x_0) + f'(x_0)(y - x_0)$.

d. Assume $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in I$. Let $z := \lambda x + (1 - \lambda)y$.

Show that (i) $f(x) \geq f(z) + f'(z)(x - z)$ and (ii) $f(y) \geq f(z) + f'(z)(y - z)$.

e. By multiplication of (i) by λ and (ii) by $(1 - \lambda)$ in item d., and adding both resulting inequalities show that if $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in I$, f is convex on I .

EXERCISE 1.39 (*Taylor's Theorem*)

Assume $f(x)$ is n times differentiable on the open interval I and $a \in I$. Below, in item d., you are asked to show that for any $x \in I$ there exists a point ξ located between x and a such that

$$f(x) = p_{n-1}(x) + \frac{f^{(n)}(\xi)}{n!}(x - a)^n, \quad (1.1)$$

where $p_{n-1}(x)$ is given by next polynomial of degree $n - 1$

$$p_{n-1}(x) := f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{(n-1)}.$$

a. Let $F(y) := f(x) - \left(f(y) + \frac{f'(y)}{1!}(x - y) + \frac{f''(y)}{2!}(x - y)^2 + \cdots + \frac{f^{(n-1)}(y)}{(n-1)!}(x - y)^{(n-1)} \right)$.

Consider $g(y) := F(y) - \left(\frac{x-y}{x-a} \right)^n F(a)$. Show that $g(a) = g(x) = 0$.

b. Use Rolle's Theorem (Exercise 1.35) to conclude that there exists a point ξ between x and a such that $F'(\xi) + \frac{n(x-\xi)^{n-1}}{(x-a)^n} F(a) = 0$.

c. Show that $F'(y) = -\frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$.

d. Calculate $F(a)$ from item b. Next show that there exists a point ξ between x and a such that (1.1) holds.

EXERCISE 1.40 (*Brouwer's Fixed Point Theorem*)

Consider a continuous function f on the compact interval $[a, b]$. Assume $f(x) \in [a, b]$ for all $x \in [a, b]$. Then Brouwer's Fixed Point Theorem states that under these assumptions, there exists a point $c \in [a, b]$ for which $f(c) = c$. (Such a point is called a *fixed point* of f). Below you are asked to prove this theorem.

a. Consider the function $g(x) = f(x) - x$. Motivate that g is continuous.

- b. Show that in case neither $f(a) = a$ or $f(b) = b$ there exists a point $c_1 \in [a, b]$ where $g(c_1) > 0$ and a point $c_2 \in [a, b]$ where $g(c_2) < 0$.
- c. Use items a and b to conclude that there must exist a point $c \in [a, b]$ for which $f(c) = c$.
- d. Consider the function $f(x) = e^{-x^3-x^2-x}$ on $[0, 1]$. Motivate why there exists a point $c \in [0, 1]$ such that $f(c) = c$.

EXERCISE 1.41 (*Fixed Point Contraction Mapping Theorem*)

Consider the function f on the closed interval I . Assume $f(x) \in I$ for all $x \in I$ and, moreover, there exists a constant $0 \leq c < 1$ such that for all $x, y \in I$, $|f(x) - f(y)| \leq c|x - y|$. Then the Contraction Mapping Theorem states that under these conditions f has a *unique* fixed point $\bar{x} \in I$. Below you are asked to show the correctness of this statement under the simplifying condition that f is continuous and the interval is compact.

- a. Assume f is continuous and I is compact. Show that f has a fixed point on I .
- b. Show that f has a unique fixed point on I . Hint: assume there is another one.
- c. Let $x \in I$. Consider the sequence defined by: $s_1 = x$ and $s_{n+1} = f(s_n)$, $n = 2, 3, \dots$. Show that $|f(s_n) - s_n| \leq c^n |s_2 - s_1|$. (Using this, one can show that this implies that f must have a fixed point, i.e., that the Contraction Mapping Theorem applies also without assuming continuity of f and the assumption that the interval is compact.)
- d. Consider the function $f(x) = \frac{1}{7}e^{-x^3-x^2-x}$ on $[0, 1]$. Show that f has a unique fixed point on $[0, 1]$. Hint: use the Mean Value Theorem to estimate $f(x) - f(y)$.
- e. Consider $f(x) = \frac{1}{x^2+1}$. Show that $f(x)$ has a unique fixed point. Hint: use the Mean Value Theorem to estimate $f(x) - f(y)$.

EXERCISE 1.42 (*Stationary point of a convex function is a global minimum*)

Taylor's theorem (see Exercise 1.39) states that if a function is twice differentiable we can find for every x and a , a point ζ that is located somewhere between x and a such that,

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(\zeta)(x - a)^2. \quad (1.2)$$

We will use this property to show that if a function f is convex, twice differentiable and has a stationary point at $x = a$, $f(a)$ is a global minimum.

- a. Show that if f has a stationary point at $x = a$ there exists a point ζ , that is located somewhere between x and a , such that $f(x) = f(a) + \frac{1}{2}f''(\zeta)(x - a)^2$.
- b. Use item a to show that if f is convex, $f(x) \geq f(a)$.
- c. Show that $f(x) = x^2 - 2x + e^{-x} + 1$ has a global minimum. Hint: consider the sign scheme of the derivative to conclude that f has a stationary point.

EXERCISE 1.43 Make a Taylor series expansion (1.2) for the functions f below, with $a = 0$ and $a = 1$, respectively.

a. $f(x) = e^x$.

b. $f(x) = \frac{1}{1+x}$.

c. $f(x) = x^2 + x + 2$.

EXERCISE 1.44 (*Newton-Raphson's algorithm to find a root of an equation*)

Assume a function f is twice differentiable and has a root at $x = a$. In this exercise we describe an algorithm, introduced by Newton and Raphson, to produce successively better approximations of this root a .

To that end recall from Taylor's theorem (Exercise 1.39) that for every x we can find a point ζ , located somewhere between x and a , such that:

$$f(a) = f(x) + f'(x)(a - x) + \frac{1}{2}f''(\zeta)(a - x)^2.$$

- Show that if x_n is an initial guess for the root a there exists a point ξ_n , that is located somewhere between x_n and a , such that $0 = f(x_n) + f'(x_n)(a - x_n) + \frac{1}{2}f''(\xi_n)(a - x_n)^2$.
- Assume our initial guess of the root, x_n , is already close to a , $f'(x_n) \neq 0$ and $f''(x)$ is bounded by some number M . Show that $a \approx x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$.
- Consider next successive (so-called Newton-Raphson) approximations of the root: $x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$, with $x_0 = \bar{x}$. Interpret the new guess x_{n+1} geometrically. Illustrate geometrically x_1 and x_2 under the assumptions of item b.
- Consider the error between the n^{th} approximation x_n and the root a , $\epsilon_n := x_n - a$. Show that $\epsilon_{n+1} = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} \epsilon_n^2$, with $\epsilon_0 = \bar{x} - a$.
- Consider a twice differentiable function f on the interval $[c, d]$, where $|f'(x)| \geq m > 0$ and $|f''(x)| \leq M$. Assume f has a root on this interval. Let $L := \frac{M}{2m}$. Show that with $\tilde{\epsilon}_n := L\epsilon_n$, $\tilde{\epsilon}_{n+1} \leq \tilde{\epsilon}_n^2$. Use this inequality to conclude that $\tilde{\epsilon}_{n+1} \leq \tilde{\epsilon}_0^{2^n}$. Assume $L(d - c) < 1$. Show that the Newton-Raphson algorithm converges to the unique root in the interval (c, d) .
- Assume f is 3 times differentiable on the interval (c, d) and has an extremal point in this interval. Use the Newton-Raphson procedure to derive a numerical algorithm to find this extremal point. Formulate sufficient conditions under which the algorithm will be guaranteed successful in finding this extremal point.
- Formulate a recursive Newton-Raphson procedure that will provide the location x at which the function $g(x) = e^x + x^2$ attains the value 3.

EXERCISE 1.45 (*Some important limiting behaviour of standard functions*)

- Make a Taylor series expansion for $f(x) = \ln(1 + x)$, with $a = 0$, to show that $\frac{\ln(1+x)}{x}$ approaches 1 if x approaches zero.
- Use item a. to conclude that $(1 + x)^{\frac{1}{x}}$ approaches e if x approaches zero. (Hint: use the fact that $f(x) = e^x$ is a continuous function).

- c. Make a Taylor series expansion for $f(x) = e^x$, with $a = 0$, to show that $\frac{e^x - 1}{x}$ approaches 1 if x approaches zero.
- d. Make a Taylor series expansion for $f(x) = e^x$, with $a = 0$, to show that $\frac{x^2}{e^x}$ approaches 0 if x goes to infinity. What about $\frac{x^n}{e^x}$ if x goes to infinity, if n is an arbitrary number?

EXERCISE 1.46 (*Derivative inverse function*)

Assume f is a differentiable function that is, moreover, invertible. Denote this inverse function by f^{inv} .

- a. Show that $f^{inv} \circ f(x) = x$.
- b. Show that $(f^{inv})'(y) = \frac{1}{f'(x)}$, where $y = f(x)$.
- c. $f(x) = e^{2x}$. Calculate $(f^{inv})'(1)$.

EXERCISE 1.47 (*minimum function*)

Assume f and g are continuous functions on $\mathbb{R}$. Consider the function $m(x) = \text{minimum}\{f(x), g(x)\}$ for all $x \in \mathbb{R}$. Motivate why $m(x)$ attains both a global maximum and a global minimum on $[-1, 1]$.

EXERCISE 1.48 (*characterization extremal points if derivatives up to order $n - 1$ are zero*)

Assume all derivatives of f up to some order $n - 1$ are zero at $x = a$, i.e. $f^{(k)}(a) = 0$, $k = 1, \dots, n - 1$, but $f^{(n)}(a) \neq 0$ and, moreover, $f^{(n)}(x)$ is continuous.

- a. Use Taylor's Theorem to conclude that for all x there exists a ξ between x and a such that

$$f(x) = f(a) + \frac{f^{(n)}(\xi)}{n!}(x - a)^n.$$

- b. Assume $f^{(n)}(a) > 0$. Motivate why there is an interval I , with $a \in I$, such that for all $x \in I$, $f^{(n)}(x) > 0$.
- c. Conclude from items a. and b. that if n is even and $f^{(n)}(a) > 0$, f has a minimum at $x = a$.
- d. Proceed similarly as in item c. to conclude that if n is even and $f^{(n)}(a) < 0$, f has a maximum at $x = a$.
- e. Proceed similarly as in item c. to conclude that if n is odd and $f^{(n)}(a) \neq 0$, f has neither a maximum nor a minimum at $x = a$.

EXERCISE 1.49 (*a sufficient condition under which a function is locally convex at minimum location*)

Assume $f(x)$ has a minimum at $x = a$; all derivatives of f up to some order $n - 1$ are zero at $x = a$, i.e. $f^{(k)}(a) = 0$, $k = 1, \dots, n - 1$; $f^{(n)}(a) \neq 0$; and $f^{(n)}(x)$ is continuous.

- a. Use Exercise 1.48 to conclude that n must be even and $f^{(n)}(a) > 0$.
- b. Use Taylor's Theorem to conclude that for all x there exists a ξ between x and a such that

$$f''(x) = \frac{f^{(n)}(\xi)}{(n - 2)!}(x - a)^{n-2}.$$

- c. Assume $f^{(n)}(a) > 0$. Motivate why there is an interval I , with $a \in I$, such that for all $x \in I$, $f^{(n)}(x) > 0$.
- d. Use the previous items to conclude that $f''(x) > 0$ on an open interval I containing $x = a$.
- e. Use item d. to conclude that f is locally convex at $x = a$.
- f. Compare this result with Example 1.50. Motivate which property does probably not apply in that example.

Exercises with an Economic Content

EXERCISE 1.50 The yield of corn C (in 10 kilograms) as a function of the amount of fertilizer x (in kilograms) used is given by: $R(x) = 1 + 2x - 0.025x^2$. Assume the price of corn is 2 and the price of fertilizer is $\frac{1}{2}$. Calculate the maximal profit and optimal amount of fertilizer for the farmer.

EXERCISE 1.51 The total cost of producing q PC's are $0.01q^2 + 100q + 100$. Determine the number of PC's for which the average cost are minimized.

EXERCISE 1.52 (*marginal output rule under perfect competition*)

Below you are asked to show that under perfect competition (that is, price p for the good is given for every firm) the amount of produced products q^* is found as that amount where the marginal revenues (i.e. the derivative of the revenue function) equals this price p . Furthermore, at q^* also the average cost and marginal cost coincide.

Assume the total cost of producing q units of a product are $f(q)$ and the market price of the product is p .

- a. Show that profits are maximized at q^* , where $p = f'(q^*)$.
- b. Let $AC(q)$ be the average cost of producing q units of the product. Show that the number of units for which $AC(q)$ is minimal is also found as the number of units for which marginal cost and average cost coincide.

EXERCISE 1.53 Suppose that the market price of fish depends on the quantity of fish q supplied to the fish market according to $p(q) = 100e^{-q}$. The fisherman incurs a fixed cost c . Determine the value of c at which the fisherman will stop fishing.

EXERCISE 1.54 The number of shirts produced by a textile company is $50\sqrt{L}$, where L is the amount of labour used. The price the company gets per shirt is 10 euro, the price per unit of labor is 5 euro.

- a. Determine the number of employed labour for which profits are maximal.
- b. For which price of labour will the company produce no shirts anymore?

EXERCISE 1.55 Government wants to set a tax rate, x , which maximizes total utility from taxes given by: $U(T) = T + \frac{1}{T}$. The tax, T , a person pays on gross income, W , equals $T = x * (W - 20.000)$. Determine the optimal tax rate.

EXERCISE 1.56 The cost involved for raising a dyke are $f(x) = c_1 + c_2x + c_3e^{-\alpha x}$. Here $c_1 + c_2x$ are the cost for raising the height by x metres and $c_3e^{-\alpha x}$ is an estimate of the cost when flooding occurs. Assume $\frac{c_2}{\alpha c_3} < 1$. Determine the height that minimizes the cost.

EXERCISE 1.57 Assume the cost of producing y units of a product are $y^3 - 18y^2 + 27y$ and the price per sold unit is p . Determine the supply curve of this producer as a function of p .

EXERCISE 1.58 Consider the Cobb-Douglas production function $Y(t) = K^\alpha(t)L^{1-\alpha}(t)$. Here $Y(t)$ is the quantity of the product produced using a capital input of $K(t)$ and labour input of $L(t)$, all at time t . Assume capital decreases over time and labour increases over time according the next formulae: $K'(t) = -\delta K(t)$ and $L'(t) = \eta L(t)$. Let $y(t) := \log(\frac{Y(t)}{L(t)})$. Show that $y'(t) = -\frac{1}{\ln 10} \alpha(\eta + \delta)$.

EXERCISE 1.59 A producer has next cost function $C(x) = \min\{x^2, x\}$ on $[0, 2]$, for the production of x units of a product.

- Motivate why C is continuous.
- Draw the graph of C .
- Is C differentiable?
- Calculate the maximum profit if the price per sold unit is p .
- For which price will the producer stop the production of this product?

EXERCISE 1.60 (*properties per capita instantaneous rate of growth*)

The *per capita instantaneous rate of growth* of a differentiable function $N(x)$ is defined as $r(N(x)) := \frac{N'(x)}{N(x)}$.

- Show that $r(x) = \frac{d \ln(N(x))}{dx}$.
- Show that $r(N(x)M(x)) = r(N(x)) + r(M(x))$.

EXERCISE 1.61 Assume the value of a tree (in terms of wood production) is approximately given by $R(t) = e^{\sqrt{t}}$ (t in years).

- Assuming a discount rate of r , calculate its present value.
- Calculate the year from now when it is optimal to cut the tree.

CHAPTER 2

Static Optimization: The Multi-Variable Case

2.1 Functions of more than one variable

Thus far, we considered only functions of a single variable $y = f(x)$. But the concept of a function can be readily extended to the case of two or more variables. Consider, for example, the production of a product where output is determined by the amounts of capital (K) and labor (L) employed. Then the output Q can be seen as a function of these two variables K and L , i.e., $Q = f(K, L)$. That is, for every admissible pair (K, L) there is a unique level of output assigned. The rule f thus assigns to every admissible point in a two-dimensional space a unique point in $\mathbb{R}$. Therefore, similar to the one-variable case, we define:

DEFINITION 2.1 A function f from a subset D of $\mathbb{R}^n$ to $\mathbb{R}$ (write $f : D \rightarrow \mathbb{R}$) defines a rule which assigns to each $x \in D$ a unique element $y \in \mathbb{R}$. The element y is called the *image* or *value* of the vector x and we write $y = f(x)$.

The set D is called the *domain* of f and the set $f(D) := \{f(x) \mid x \in D\}$ the *range* of f .

EXAMPLE 2.2 Consider $f(x, y) = x + y$, for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This rule defines a function from $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ to $\mathbb{R}$. This, since to every (x, y) in this domain the unique element $x + y$ is assigned. The range of f is the interval $[0, 2]$.

Note that in this example most of the elements in the range are attained more than once.

EXERCISE 2.1 Determine in the exercises below a set D as large as possible such that the rule f defines a function from $D \rightarrow \mathbb{R}$. Calculate the range of the corresponding function.

a. $f(x, y) = x + y$.

c. $f(x, y, z) = e^{x-y} + \frac{1}{z^2}$.

b. $f(x, y) = x^2 + y^2$.

d. $f(x, y) = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y \geq 0\}$.

If the domain of a function f is a subset of $\mathbb{R}^n$, the *graph* of the function $f(x)$ is the ordered set $(x, f(x))$ in the $(n + 1)$ -dimensional space. If $n = 2$, in most cases, it is very difficult to visualize the graph of a function. In case n is larger than 4 it is even impossible to visualize it. Nonetheless, we call the locus of these points the *hypersurface* or *graph* of f . In Figure 2.1 we visualized the graph of the function from Example 2.2.

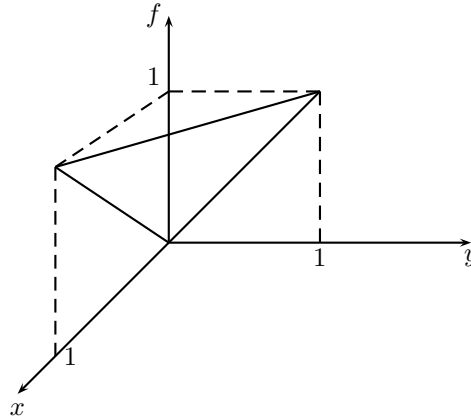


FIGURE 2.1 – Graph of $f(x, y) = x + y$.

Functions of more than one variable can be classified into various types. Two well-known types are:

A. Affine functions: $f(x_1, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n + b$, where b and a_i are scalars.

Geometrically, if $n = 2$, the graph of an affine function represents a plane, where b is the value of the function at the origin $(0, \dots, 0)$. In case $b = 0$, the function is usually called a *linear* function.

B. Quadratic functions:

A quadratic function involves functions where variables occur with a degree at most two, and the sum of exponents of the variables appearing in any single term does not exceed two. An example of a quadratic function is:

$$f(x_1, x_2) = b + a_1x_1 + a_2x_2 + c_1x_1^2 + c_2x_2^2 + d_1x_1x_2.$$

In case the constant and linear terms are lacking (i.e., in the above example $b = a_i = 0$), the function is usually called a *quadratic form*. We will provide an easy characterization of a quadratic form later on.

EXERCISE 2.2 Classify which of the next functions is linear, affine, quadratic or a quadratic form.

a. $f(x, y) = 2x + y + 1$.

c. $f(x, y, z) = 1 + x + y + z + xz$.

b. $f(x, y) = x^2 + y^2$.

d. $f(x, y, z) = x + 2y - 3z$.

To visualize (or at least get an idea about) the graph of a function of more than one variable one often considers *intersection curves* of the graph. Assume $\phi(\cdot)$ represents a curve in the domain of the

function f parameterized by the "time" variable t , that is, $\phi(t)$ is the point the curve reaches at "time" t . Then the intersection curve of f with respect to $\phi(\cdot)$ is the curve $(\phi(t), f(\phi(t)))$. That is the curve obtained by a vertical intersection of the graph of f at every point $\phi(t)$ in the domain. Figure 2.2 illustrates.

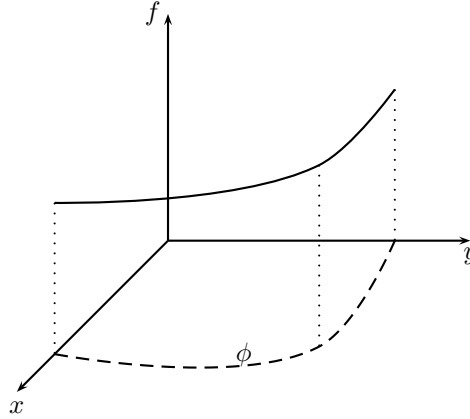


FIGURE 2.2 – Intersection curve of $f(x, y)$ with respect to $\phi(\cdot)$.

Particularly useful in this respect are intersection curves of f with respect to lines. That is, if x, y are two points in the domain of f , the intersection of the graph of f with the vertical plane through x and y . In case one takes all but one of the variables constant, the function reduces to a function of just one variable. Such an intersection curve is called a *section function*. Geometrically, in case we consider a function $f(x, y)$ of two variables, a section function is the intersection of the graph of f at a point with a vertical plane in either the x -direction or the y -direction. For instance, if $f(x, y) = x + y$, the section curve at the point $(1, 2)$ in the y -direction is the intersection curve of the vertical plane $x = 1$ with the graph of f . This is given by $g(y) = 1 + y$. Geometrically, this section function is obtained by intersecting the graph of f with a vertical plane parallel to the y -axis at $x = 1$, see Figure 2.3.

Similarly, one can consider in $(1, 2)$ the section curve in the x -direction with the graph of f . This section curve is given by $h(x) = x + 2$. Geometrically, this section function is obtained by intersecting the graph of f with a vertical plane parallel to the x -axis at $y = 2$, see Figure 2.4.

Another way to get insight into the graph of a two dimensional function $f(x, y)$ is by plotting *level curves (or isoclines)* of it. That is, by plotting the points in the (x, y) -plane that provide the same function value (e.g, a value of 3). Geometrically, a level curve is obtained as the intersection of the graph of f with a horizontal plane at a fixed height (the considered level, e.g., 3). So, for $f(x, y) = x + y$, the level curve corresponding with a level of 3, are all those points in the (x, y) -plane that satisfy the relation $x + y = 3$. Particularly in economics isoclines are often used. E.g., if we consider a production function $Q = f(K, L)$, then an iso-production curve provides all combinations of capital (K) and labor (L) that produce the same output. Figure 2.5 illustrates the idea.

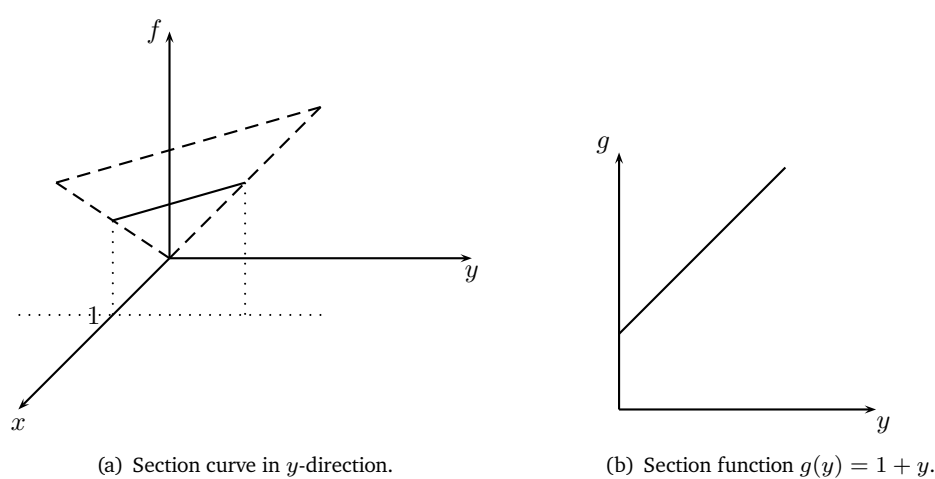


FIGURE 2.3 – Section curve $f(x, y) = x + y$ at $x = 1$.

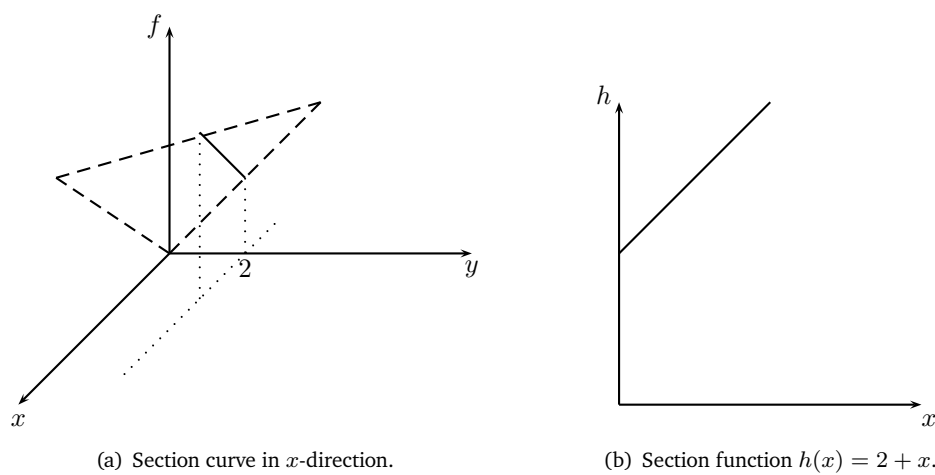


FIGURE 2.4 – Section curve $f(x, y) = x + y$ at $y = 2$.

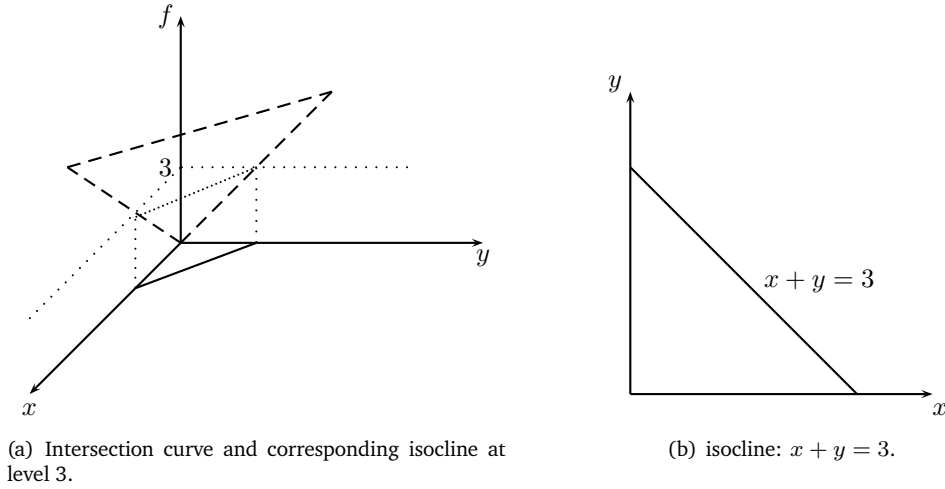


FIGURE 2.5 – Isocline of $f(x, y) = x + y$ at a level of 3.

EXERCISE 2.3 For the functions $f(x, y)$, below, calculate the section curve at the point $(3, 4)$ in the x -direction and y -direction, respectively. Also, calculate the level curve corresponding with a level of 4. Plot all these curves in the (x, y) -plane.

- a. $f(x, y) = 3x - 4y + 1$. c. $f(x, y) = 2^{xy}$. e. $f(x, y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$, $x \geq 0, y \geq 0$.
 b. $f(x, y) = x^2 + y^2$. d. $f(x, y) = -\ln(x + y)$, $x + y > 0$.

2.2 Distance, Bounded Sets and Compact Sets

We want to generalize the notion of compact set for the scalar case to sets consisting of more than one variable. As an example we consider the set of all nonnegative points in $\mathbb{R}^2$, i.e., $S := \{(x, y) \mid x \geq 0, y \geq 0\}$ and ask ourselves the question whether this set S is compact. Recall from the previous chapter that we called a set in $\mathbb{R}$ compact if it is both bounded and closed. We need to generalize each of these notions. Let us begin with boundedness.

Clearly, we would not want to describe The set S mentioned above as bounded. But why is that the case? Upon closer reflection, the reason is that the distance between points in this set can become arbitrarily large. That is, given any two points in S , say (a_1, b_1) and (a_2, b_2) separated by some distance m , it is always possible to find another point (a_3, b_3) whose distance from (a_1, b_1) is much greater than m .

This observation motivates the definition of boundedness: a set S is called bounded if there exists a circle that contains all points of S . More generally, in $\mathbb{R}^2$, we say that a set S is bounded if there exists a circle centered at $(0, 0)$ with some finite radius that contains the entire set.

In terms of distances, the interior of a circle (or disc) is characterized by the property that all points within it lie at a distance from the center that does not exceed a certain number r . For a point (x, y) ,

the distance to the origin $(0, 0)$ is given by $\sqrt{x^2 + y^2}$. More formally, the distance from (x, y) to the origin is defined as the length of the line segment connecting (x, y) to the origin. This is illustrated in Figure 2.6.

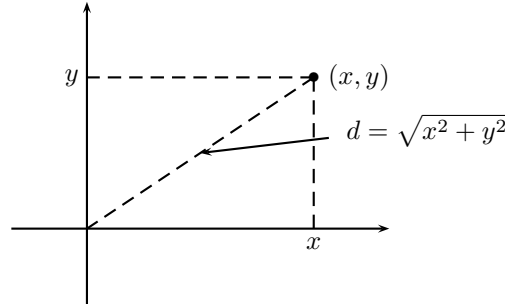


FIGURE 2.6 – Distance d of (x, y) to origin.

In $\mathbb{R}^3$ the distance d from a point (x, y, z) to the origin $(0, 0, 0)$ is given by $d = \sqrt{x^2 + y^2 + z^2}$. This definition extends straightforwardly to higher dimensions. In general, the distance from a point $(x_1, \dots, x_n)$ to the origin $(0, \dots, 0)$ is defined as $\sqrt{x_1^2 + \dots + x_n^2}$. We denote this distance using the shorthand notation $\|x\|$. More generally, we define the distance between two points x and $y \in \mathbb{R}^n$ as the length of the line segment connecting x and y or, equivalently, as the distance from the point $(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$ to the origin, that is: $\|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$.

A ball in $\mathbb{R}^n$ with radius r is then defined as the set of all points in $\mathbb{R}^n$ whose distance to the origin is smaller than r . In $\mathbb{R}^3$, this corresponds to the set of all points lying inside a sphere centered at $(0, 0, 0)$ with radius r . Figure 2.7 illustrates the idea for $n = 2$, in which case a "ball" reduces to a circle with center $(0, 0)$ and radius r . Using this idea, we next generalize the notion of a bounded set as follows.

DEFINITION 2.3 A set $S \subset \mathbb{R}^n$ is called *bounded* if there exists a ball with radius r such that all $x \in S$ belong to this ball.

Figure 2.7 shows that the set S is bounded, since there exists a "ball" that completely contains S .

We now turn to generalizing the notions of openness and closedness to subsets of $\mathbb{R}^n$.

The underlying ideas closely parallel those introduced earlier for subsets of $\mathbb{R}$. For example, a set S is called closed if it has the property that, whenever a sequence⁵ $\{s_n\}_{n=1}^{\infty}$ of points in S converges, its limit s also belongs to S .

Similarly, a set S is called open if for every point $s \in S$ there exists a radius $r > 0$ such that the ball centered at s with radius r is entirely contained in S .

⁵ More precisely, a sequence $\{s_n\}_{n=1}^{\infty}$ is a set of points indexed by a number n : s_1, s_2, s_3 etc.

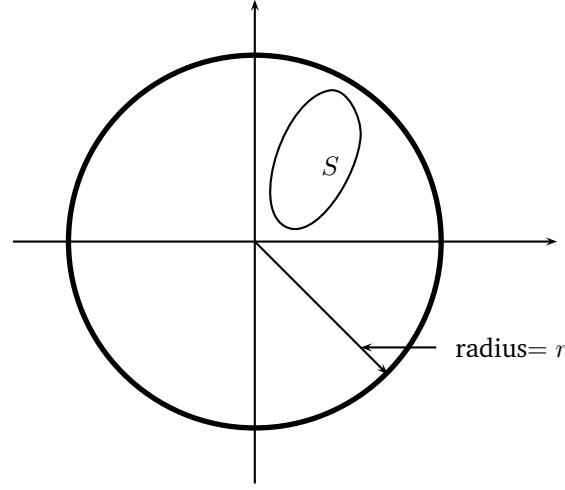


FIGURE 2.7 – Bounded set S .

DEFINITION 2.4 A set $S \subset \mathbb{R}^n$ is called *open* if for every $x \in S$ one can find a ball with center x and positive radius, such that all elements of this ball are an element of S .

The set S is called *closed* if the complement set of S is open. Or, equivalently, if for every sequence of points that has a limit, with all points belonging to S , the limit of this sequence also belongs to S .

From the definition of closedness it is clear that this can also be formalized again using the next generalization of the notions of boundary points and accumulation points.

DEFINITION 2.5 A point $b \in S$ is called a *boundary point* of the set S if every ball with center b contains at least one point not belonging to S . Or, equivalently, if there exists a sequence of numbers, all not belonging to S , that converges to b .

A point a is called an *accumulation point* of the set S if every ball with center a contains at least one point in S , different from a , and one point outside S . Or, equivalently, if there exists both a sequence of numbers $\{s_n\}_{n=1}^{\infty}$, all not belonging to S , that converges to a , and another sequence of numbers $\{t_n\}_{n=1}^{\infty}$ (where not all numbers coincide with a from a certain index on), that all do belong to S , that converges to a .

From these definitions it follows now directly that a set S is closed if and only if S contains all its accumulation points. Or, stated differently, all accumulation points of S are boundary points.

EXAMPLE 2.6 Consider the set $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$. Then it is clear that for every point $(\bar{x}, \bar{y}) \in S$ there is a circle with center $(\bar{x}, \bar{y})$ and some radius $\bar{r}$ such that the whole circle lies completely in S . Therefore, S is open. Figure 2.8 illustrates.

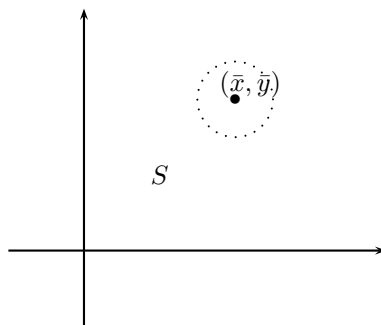


FIGURE 2.8 – Open set $S = \{(x, y) \mid x > 0, y > 0\}$.

EXAMPLE 2.7 Consider the set $S = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1\}$. Then it is clear that whatever sequence of points we consider in S , and this sequence converges to some limit, this limit also belongs to S . Figure 2.9 illustrates this case.

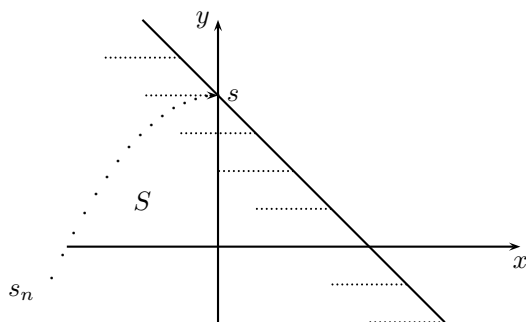


FIGURE 2.9 – $S = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1\}$ is closed.

As in the scalar case, finite unions and intersections of closed (open) sets in $\mathbb{R}^n$ are themselves closed (open). We will make extensive use of this property later when determining whether particular sets are closed.

PROPOSITION 2.8 If S and T are two closed (open) sets, then $S \cap T$ and $S \cup T$ are closed (open) too.

Using the above definitions we can generalize now the notion of a compact set.

DEFINITION 2.9 A set $S \subset \mathbb{R}^n$ is called *compact* if it is both bounded and closed.

EXAMPLE 2.10 Consider the unit box, S , in $\mathbb{R}^2$. That is, $S = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } |y| \leq 1\}$. S is clearly bounded (consider, e.g., a circle with radius $\sqrt{3}$). Furthermore, S is the intersection of the sets $\{(x, y) \in \mathbb{R}^2 \mid x \leq 1\}$, $\{(x, y) \in \mathbb{R}^2 \mid x \geq -1\}$, $\{(x, y) \in \mathbb{R}^2 \mid y \leq 1\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y \geq -1\}$. All these sets are closed, so, by Proposition 2.8, S is closed too. Consequently the unit box is a compact set. Figure 2.10 illustrates this example.

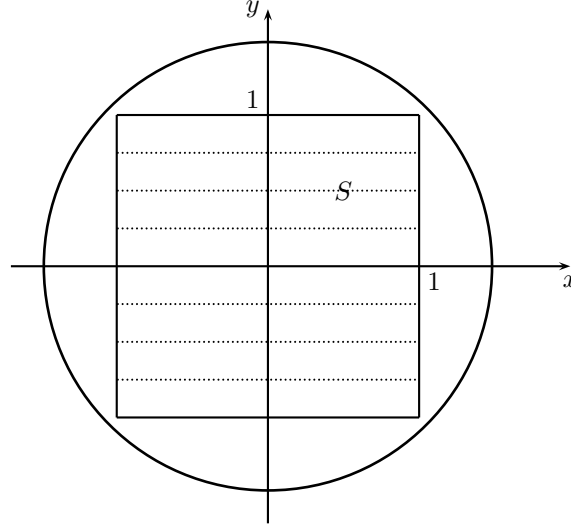


FIGURE 2.10 – unit box, S , is compact.

EXERCISE 2.4 Determine the boundary and accumulation points of next sets. Motivate which of these sets is bounded, open, closed or compact, respectively.

- a. $\{(x, y) \in \mathbb{R}^2 \mid x \geq 2\}$.
- b. $\{(x, y) \in \mathbb{R}^2 \mid y \geq x \text{ and } y \leq x + 1\}$.
- c. $\{(x, y) \in \mathbb{R}^2 \mid |x| \leq 2, |y| < 2\}$.
- d. $\{(1, 2)\}$.
- e. $\{(x, y) \in \mathbb{R}^2 \mid x < 1, x \geq -1, y \leq -x, y \geq x\}$.

2.3 Continuous Functions

For scalar functions we introduced in Chapter 1 intuitively the notion of continuity. The basic idea was to call a function continuous at a point x_0 if its graph does not exhibit a "jump" or "break" above that point. Roughly speaking, this means that the graph is connected in an open interval containing x_0 , so that it can be drawn without lifting one's pencil from the paper. Or, stated differently, the function f has the property that, in case we approach the point x_0 in the domain with some sequence $\{s_n\}_{n=1}^{\infty}$, the corresponding sequence of function values $\{f(s_n)\}_{n=1}^{\infty}$ approach the value of the function at x_0 , i.e. $f(x_0)$. We extend this idea to define continuity for functions of several variables. A function

f defined on an open set is called continuous if its graph is "connected" over that set. Figure 2.11 illustrates the concept of this "limiting behavior". Two sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are drawn together with their function values $\{f(s_n)\}_{n=1}^{\infty}$ and $\{f(t_n)\}_{n=1}^{\infty}$, respectively. The graph illustrates that if $\{s_n\}_{n=1}^{\infty}$ converges to (x_0, y_0) , $\{f(s_n)\}_{n=1}^{\infty}$ converges to $f(x_0, y_0)$. Similarly this applies for $\{f(t_n)\}_{n=1}^{\infty}$. If this property holds for every sequence that converges to (x_0, y_0) , f is called continuous at (x_0, y_0) .

DEFINITION 2.11 Let the function f be defined on an open set S and $\bar{s} \in S$. Then f is called *continuous* at $\bar{s}$, in case for every sequence $\{s_n\}_{n=1}^{\infty}$ that approaches $\bar{s}$ it holds that $\{f(s_n)\}_{n=1}^{\infty}$ approaches $f(\bar{s})$. If f is continuous at every $\bar{s} \in S$, f is called continuous on S . If f is not continuous it is called *discontinuous*.

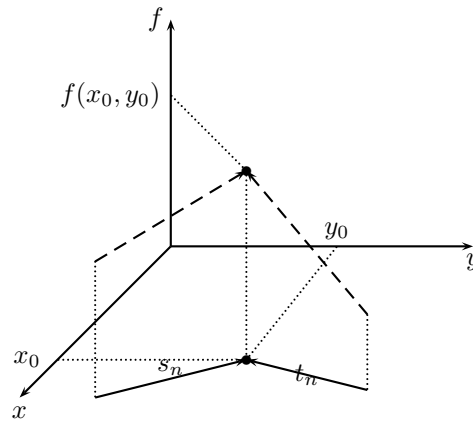


FIGURE 2.11 – Continuity of f at (x_0, y_0) .

EXERCISE 2.5 Consider the function $f(x, y) = \begin{cases} 0 & \text{if } x \geq 0; \\ 1 & \text{if } x < 0. \end{cases}$

- Show that the sequence $\{s_n\}_{n=1}^{\infty}$ with $s_n = (\frac{-1}{n}, 0)$, $n = 1, 2, \dots$ converges to $(0, 0)$.
- Show that f is not continuous in $(0, 0)$.

EXERCISE 2.6 Show that the function $f(x, y)$ defined by $f(x, y) = \begin{cases} \frac{1}{x+y} & \text{if } x+y \neq 0; \\ 0 & \text{if } x+y = 0; \end{cases}$ is not continuous in $(0, 0)$.

Using the "limiting behavior" idea it can be easily shown that similar to the scalar case we have the next results.

THEOREM 2.12 Let f and g be continuous functions on a set S , and λ, μ be real numbers. Then the following functions are continuous on S too:

$$i) \lambda f + \mu g; \quad ii) fg; \quad iii) \frac{f}{g}, \text{ provided that } g(x) \neq 0 \text{ for all } x \in S.$$

Furthermore, if f is continuous on $g(S)$ and g is continuous on S , also the composite function $f \circ g$ is continuous on S .

By relying on the continuity properties of scalar functions such as polynomials, rational functions, exponential- and logarithmic functions, we can directly conclude from Theorem 2.12, that sums, products and quotients of multi-variable functions of this type are also continuous, provided the denominator is nonzero.

EXAMPLE 2.13 Using Theorem 2.12 one can easily verify that next two functions are continuous.

1. The function $f(x, y) = \frac{x^2+y}{x^2+y^2+1}$ is continuous. This is because:
 - i) $f_1(x, y) = x^2$ and $f_2(x, y) = y$ are continuous, so $g_1(x, y) = x^2 + y$ is continuous;
 - ii) $f_1(x, y)$, $f_3(x, y) = y^2$ and $f_4(x, y) = 1$ are continuous, so $g_2(x, y) = x^2 + y^2 + 1$ is continuous;
 - iii) as both $g_1(x, y)$ and $g_2(x, y)$ are continuous and, moreover, $g_2(x, y) \neq 0$, $f(x, y) = \frac{g_1(x, y)}{g_2(x, y)}$ is continuous.
2. The function $f(x, y) = e^{x+y^2}$ is continuous. This is because:
 - i) $f_1(x, y) = x$ and $f_2(x, y) = y^2$ are continuous, so $g(x, y) = x + y^2$ is continuous;
 - ii) $g_2(z) = e^z$ is continuous. So, $f(x, y) = g_2(g_1(x, y))$ is continuous.

Furthermore, the continuity property for scalar functions can also be generalized. To do this, we first formalize the concept of a connected set. Definition 2.14 captures the idea that a set S is called connected if any two points in S can be joined by a curve that lies entirely within S .

DEFINITION 2.14 A set S is called (path-)connected if for any two points in S there is a path⁶ in S joining them.

The continuity property then generalizes as stated in Proposition 2.15. In Exercise 2.40, you are asked to prove the first part of this proposition.

⁶ A path from a point x to a point y in a set $S \subset \mathbb{R}^n$ is a continuous function f from the interval $[0, 1]$ to S with $f(0) = x$ and $f(1) = y$.

PROPOSITION 2.15 (*Continuity property*)

Let the continuous function f be defined on a set S . Then, if S is

1. a connected set, the set $f(S)$ is an interval.
2. compact, the set $f(S)$ is a compact set.

From Proposition 2.15 then immediately results the next important theorem in optimization.

THEOREM 2.16 (*Weierstrass' theorem*)

Let f be continuous on a compact set S . Then f attains both a global maximum value and a global minimum value on S .

EXAMPLE 2.17 Consider the function $f(x, y) = xy - ye^{x^2-4y+1} + \ln(x^2 + y^2 + 1)$ on $S = \{(x, y) \mid |x| \leq 1, |y| \leq 2\}$. By Theorem 2.12, it follows directly that f is continuous on S . Additionally, S is a compact set. Therefore, by Weierstrass' theorem, f attains both a maximum and minimum on S . However, it is not analytically possible to determine the exact locations where these maximum and minimum values occur.

EXERCISE 2.7 Motivate which of the next functions is continuous.

- a. $f(x, y) = \begin{cases} \log(x^2 + y^2 + 1), & x \leq 0; \\ \log(xy^2 + y^2 + 1), & x > 0. \end{cases}$
- b. $f(x, y, z) = ze^{x+\ln(y^2+1)}.$
- c. $f(x, y) = \begin{cases} \frac{1}{x^2+y^2}, & (x, y) \neq (0, 0); \\ 0, & (x, y) = (0, 0). \end{cases}$

EXERCISE 2.8 Motivate whether the next function has an optimum on S .

- a. $f(x, y) = x^2 + y^2$, $S = \{(x, y) \mid x > -1, y > -1\}$.
- b. $f(x, y, z) = \frac{xyz^2 - e^x}{\ln(x^2 + 2y + z)}$, $S = \{(x, y) \mid -10 \leq x \leq 12, 1 \leq y \leq 2, 2 \leq z \leq 3\}$.

By definition, a set S is called closed, if for any sequence of points chosen from S , the limit of the sequence (if it exists) also belongs to S . Now, suppose the set S consists of all points satisfying the inequality $g(x, y) \leq 0$. If $g(x, y)$ is a continuous function, it follows directly from the definition of continuity that S is closed. For, consider a sequence $\{s_n\}_{n=1}^{\infty}$, with $s_n := (x_n, y_n)$, such that $g(x_n, y_n) \leq 0$ for all n , and suppose $\{s_n\}_{n=1}^{\infty}$ converges to some limit s . By the continuity of g , the sequence $\{g(s_n)\}_{n=1}^{\infty}$ converges to $g(s)$. Since $g(s_n) \leq 0$ for every n , it follows that the limit also satisfies $g(s) \leq 0$. In other words, the limit s of the sequence $\{s_n\}_{n=1}^{\infty}$ also satisfies $g(s) \leq 0$. That is, s belongs to S too. Proposition 2.18 formalizes this result.

PROPOSITION 2.18 Consider $S = \{s \in \mathbb{R}^n \mid g(s) \leq 0\}$. Assume $g(s)$ is continuous. Then, S is closed.

EXERCISE 2.9 Motivate why the next set S is compact. $S = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4 \text{ and } xy \geq 1\}$.

2.4 Differentiable Functions

In Chapter 1 we introduced differentiability for a scalar function f at a point x_0 . Before giving a formal definition, we first discussed the intuition behind this notion. We explained that a function f is called differentiable at x_0 if its graph has a unique tangent line at that point, allowing us to accurately estimate changes in the function value near x_0 . For functions of two variables, this idea generalizes as follows. We call a function f differentiable at a point (x_0, y_0) if there exists a unique tangent plane to the graph of f at that point. Figure 2.12 illustrates this concept.

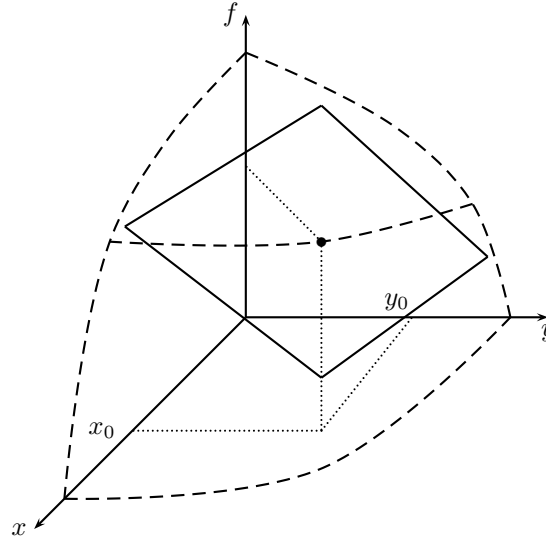


FIGURE 2.12 – f is differentiable at (x_0, y_0) .

Note that this tangent plane to the graph of f at (x_0, y_0) only exists in case both the section function $g(x) = f(x, y_0)$ has a tangent line to the graph at $x = x_0$ and the section function $h(y) = f(x_0, y)$ has a tangent line to the graph at $y = y_0$. Figure 2.13 illustrates this.

If we denote the slopes of these section functions at (x_0, y_0) by $D_1f(x_0, y_0)$ and $D_2f(x_0, y_0)$, respectively, then the plane passing through $f(x_0, y_0)$ with these slopes in the x - and y - directions is given by

$$z(x, y) = f(x_0, y_0) + D_1f(x_0, y_0)(x - x_0) + D_2f(x_0, y_0)(y - y_0). \quad (2.1)$$

As in the scalar case, we call a function "smooth" (or differentiable) at (x_0, y_0) if function values in a neighborhood of (x_0, y_0) can be accurately approximated by the corresponding values on this plane $z(x, y)$. That is, in analogy with the scalar case, the function f is differentiable at (x_0, y_0) if the ratio of the difference between $f(x, y)$ and $z(x, y)$ to the distance between (x, y) and (x_0, y_0) tends to zero

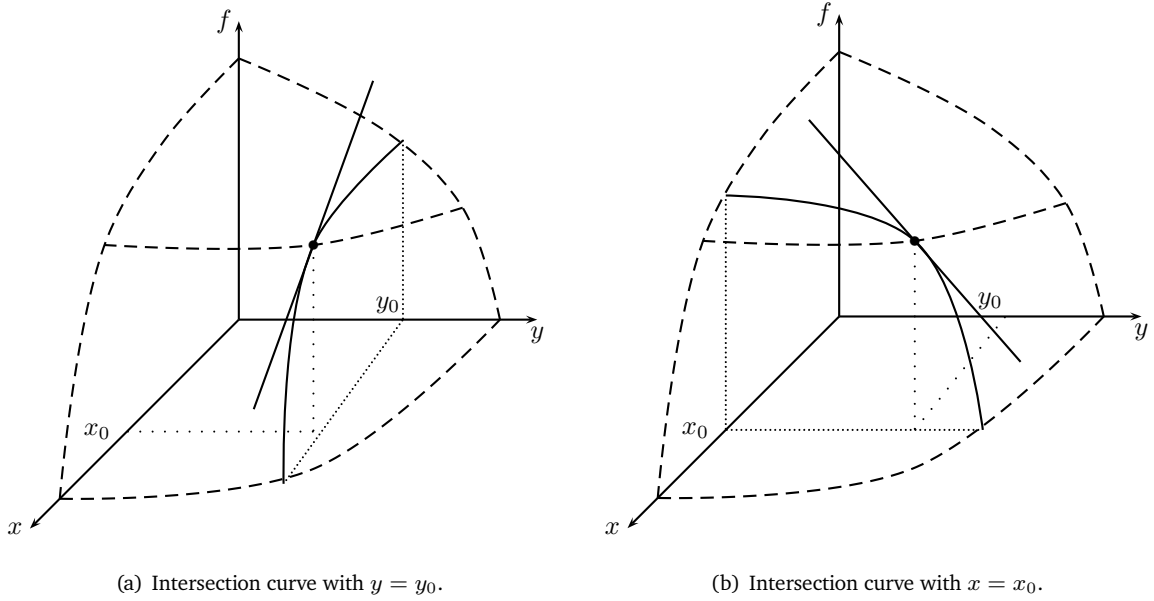


FIGURE 2.13 – Intersection curves f at (x_0, y_0) .

as (x, y) approaches (x_0, y_0) . That is if, with $z(x, y)$ as defined above,

$$\frac{f(x, y) - z(x, y)}{\|(x, y) - (x_0, y_0)\|} \text{ tends to zero if } (x, y) \text{ approaches } (x_0, y_0). \quad (2.2)$$

EXAMPLE 2.19 Consider the question whether $f(x, y) = x^2 + y^2$ is "smooth" (in the above mentioned sense) at (x_0, y_0) .

The section function $f(x, y_0) = x^2 + y_0^2$ is differentiable at $x = x_0$ and its derivative is $D_1 f(x_0, y_0) = 2x_0$. Similarly, the section function $f(x_0, y) = x_0^2 + y^2$ is differentiable at $y = y_0$ and its derivative is $D_2 f(x_0, y_0) = 2y_0$.

So the candidate tangent plane to the graph of f at (x_0, y_0) is:

$$z(x, y) = x_0^2 + y_0^2 + 2x_0(x - x_0) + 2y_0(y - y_0).$$

To verify "smoothness" of f in the above mentioned sense, we consider

$$\frac{f(x, y) - z(x, y)}{\|(x, y) - (x_0, y_0)\|}.$$

As we have to consider the limiting value of this fraction if (x, y) approaches (x_0, y_0) it is convenient to consider the difference between these two numbers $(\Delta x, \Delta y) := (x - x_0, y - y_0)$. Substitution of

this into the fraction shows we have to consider

$$\begin{aligned} & \frac{f(x_0 + \Delta x, y_0 + \Delta y) - z(x_0 + \Delta x, y_0 + \Delta y)}{\|(\Delta x, \Delta y)\|} = \\ & \frac{(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 - x_0^2 - y_0^2 - 2x_0\Delta x - 2y_0\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \\ & \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \sqrt{(\Delta x)^2 + (\Delta y)^2}. \end{aligned}$$

Clearly, the last expression goes to zero if Δx and Δy go to zero.

So, f is "smooth" at every point (x_0, y_0) of its domain.

As mentioned earlier, if the derivatives of both section functions exist at the point (x_0, y_0) , there is exactly one plane that passes through $f(x_0, y_0)$ with slopes in the x - and y - directions given by these derivatives. In that sense, the candidate tangent plane is uniquely determined. However, as the next example shows, the existence of these two derivatives alone does not necessarily guarantee that a unique tangent plane to the graph of a function exists at (x_0, y_0) .

EXAMPLE 2.20 Consider the function $f(x, y) = \begin{cases} |x|, & y \neq 0; \\ 0, & y = 0. \end{cases}$

Then at $(0, 0)$ the section function $g(x) = f(x, 0) = 0$. So, the derivative of $g(x)$ is zero at $x = 0$. Furthermore, the section function $h(y) = f(0, y) = 0$. So, the derivative of $h(y)$ is zero at $y = 0$ too. So, the candidate tangent plane is given by the horizontal plane $z = 0$. On the other hand, for every $y_0 \neq 0$, the section function in the x -direction is $g_1(x) = |x|$. Consequently, if we approach $(0, 0)$ via a line from any other direction than the x -axis or the y -axis, it is clear that the corresponding intersection curve is not differentiable. In other words, the graph of f does not have a unique tangent plane at $(0, 0)$.

As we already indicated, the slope in the x -direction of the tangent plane can be determined by considering the derivative of the section function $f(x, y_0)$ at $x = x_0$. Similarly, the slope in the y -direction of this tangent plane can be determined by considering the derivative of the section function $f(x_0, y)$ at $y = y_0$. For that reason, we will call these derivatives together "the derivative of f " if the tangent plane to the graph of f is uniquely determined at (x_0, y_0) .

Below, we summarize these observations in a number of definitions and theorems.

DEFINITION 2.21 Consider the function $f(x, y)$ at the point (x_0, y_0) . If the section function $f(x, y_0)$ is differentiable at $x = x_0$, we say that f is *partially differentiable w.r.t. x* at (x_0, y_0) . The derivative of this section function $f(x, y_0)$ at $x = x_0$ is called the partial derivative of f with respect to x . The notation for this is:

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } D_1 f(x_0, y_0) \text{ or } f'_x(x_0, y_0).$$

Similarly, if the section function $f(x_0, y)$ is differentiable at $y = y_0$, we say that f is *partially differentiable w.r.t. y* at (x_0, y_0) . The derivative of this section function $f(x_0, y)$ at $y = y_0$ is called the partial derivative of f with respect to y . The notation for this is:

$$\frac{\partial f}{\partial y}(x_0, y_0) \text{ or } D_2 f(x_0, y_0) \text{ or } f'_y(x_0, y_0).$$

If both partial derivatives exist, we say that f is partially differentiable at (x_0, y_0) .

Furthermore, the row matrix containing the partial derivatives, $[D_1 f(x_0, y_0) \ D_2 f(x_0, y_0)]$, is called the *matrix of partial derivatives* or *Jacobi matrix* at (x_0, y_0) , and is denoted by $Df(x_0, y_0)$.

EXAMPLE 2.22 Consider the function $f(x, y) = 2x^2 + y + xy$ at $(x_0, y_0) = (2, 1)$.

First, define $g(x) := f(x, 1) = 2x^2 + 1 + x$. This section function is differentiable, and in particular, at $x = 2$. Note that $g'(x) = 4x + 1$, and $g'(2) = 9$. Therefore, $\frac{\partial f}{\partial x}(2, 1) = 9$.

Similarly, define the section function $h(y) := f(2, y) = 8 + 3y$. Clearly $h(y)$ is differentiable, and $h'(1) = 3$. Therefore $\frac{\partial f}{\partial y}(2, 1) = 3$.

Thus, the Jacobi matrix of f at $(2, 1)$ is: $Df(2, 1) = [9 \ 3]$.

EXERCISE 2.10 Calculate next partial derivatives.

- a. $\frac{\partial f}{\partial x}(1, 0)$ and $\frac{\partial f}{\partial y}(1, 0)$ if $f(x, y) = xy$. b. $\frac{\partial f}{\partial x}(1, 2)$ and $\frac{\partial f}{\partial y}(1, 2)$ if $f(x, y) = \frac{y}{x}$.

DEFINITION 2.23 Suppose that f is defined on an open set S containing the point (x_0, y_0) and f is partial differentiable at (x_0, y_0) .

Consider the equation (2.1), which represents the plane $z(x, y)$ passing through $f(x_0, y_0)$ with the slope in the x -direction given by $D_1 f(x_0, y_0)$ and the slope in the y -direction by $D_2 f(x_0, y_0)$.

We say that f is *differentiable* at the point (x_0, y_0) if the quotient (2.2) approaches zero if (x, y) goes to (x_0, y_0) .

If f is differentiable at (x_0, y_0) , $z(x, y)$ is called the tangent plane to the graph of f at (x_0, y_0) .

Furthermore, the row matrix of partial derivatives $[D_1 f(x_0, y_0) \ D_2 f(x_0, y_0)] =: Df(x_0, y_0)$, is called the *derivative of f* at (x_0, y_0) and is denoted by $f'(x_0, y_0)$.

If f is differentiable at every point in S , then f is said to be differentiable on S .

EXAMPLE 2.24 In Example 2.19, $f'(2, 1) = [4 \ 2]$.

As in the case of scalar functions, it follows directly from the definition that a multi-variable differentiable function has to be continuous (see Exercise 2.42).

THEOREM 2.25 *Let f be differentiable on an open set containing (x_0, y_0) . Then f is continuous at (x_0, y_0) .*

Clearly, in practice one would prefer to avoid verifying the limit in (2.2) when determining whether a function f is differentiable. We will see shortly that most functions commonly encountered are indeed differentiable. Another important tool for establishing differentiability is the examination of a function's partial derivatives. The next result, Theorem 2.26, states this criterion.

THEOREM 2.26 *Suppose that f is defined on an open set S containing the point (x_0, y_0) . If for every point in an open neighborhood of (x_0, y_0) the partial derivatives of f exist and these partial derivatives are continuous in (x_0, y_0) , then f is differentiable at (x_0, y_0) .*

Most functions encountered in the literature possess continuous partial derivatives and are therefore differentiable. Functions whose partial derivatives are continuous are called *continuous differentiable* and are denoted by $f \in C^1$. As one might expect, there do exist differentiable functions whose partial derivatives are not continuous. Readers interested in further details are referred to the literature (see, for example, [4]). In the advanced exercises of this chapter we have also included several problems that illustrate properties related to continuity, partial differentiability and differentiability of functions.

EXAMPLE 2.27 Consider $f(x, y) = -(x^2 + y^2) + 7$. Then, $Df(x, y) = [-2x \ -2y]$. Clearly, both partial derivatives are continuous. Therefore, f is differentiable at, for example, $(2, 1)$, and $f'(2, 1) = [-4 \ -2]$.

The tangent plane to the graph of $f(x, y)$ at $(x, y) = (2, 1)$ is given by:

$$z = 2 - 4(x - 2) - 2(y - 1) = 12 - 4x - 2y.$$

Figure 2.14 illustrates this case.

EXERCISE 2.11 Use Theorem 2.26 to show that next functions are differentiable. Determine their derivative.

a. $f(x, y) = 1$.

b. $f(x, y) = x$.

c. $f(x, y) = y$.

Similarly as for scalar functions, one can show that the next property holds for differentiable functions (see also Exercise 2.51).

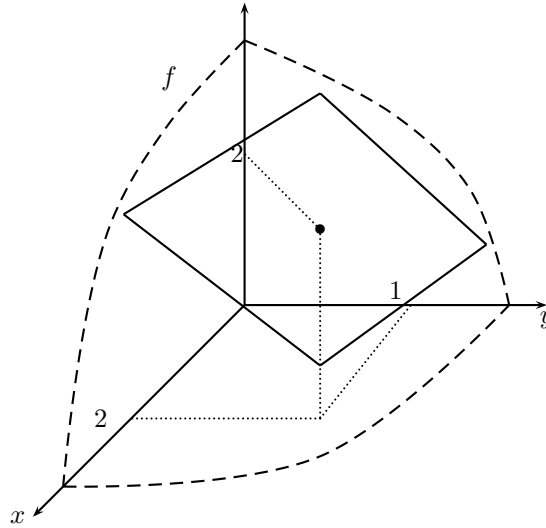


FIGURE 2.14 – Tangent plane at $(2, 1)$ to $f(x, y) = -(x^2 + y^2) + 7$.

THEOREM 2.28 Let f and g be differentiable at (x_0, y_0) , and λ, μ be real numbers. Then

- i) $\lambda f + \mu g$ is differentiable at (x_0, y_0) , and $(\lambda f + \mu g)'(x_0, y_0) = \lambda f'(x_0, y_0) + \mu g'(x_0, y_0)$;
- ii) fg is differentiable at (x_0, y_0) , and $(fg)'(x_0, y_0) = f'(x_0, y_0)g(x_0, y_0) + f(x_0, y_0)g'(x_0, y_0)$;
- iii) $\frac{f}{g}$ is differentiable at (x_0, y_0) (provided $g(x_0, y_0) \neq 0$),

$$\text{and } \left(\frac{f}{g}\right)'(x_0, y_0) = \frac{f'(x_0, y_0)g(x_0, y_0) - g'(x_0, y_0)f(x_0, y_0)}{g^2(x_0, y_0)}.$$

Furthermore, if f is differentiable at $g(x_0, y_0)$ and g is differentiable at (x_0, y_0)

- iv) $f \circ g$ is differentiable at (x_0, y_0) , and $(f \circ g)'(x_0, y_0) = f'(g(x_0, y_0))g'(x_0, y_0)$.

Item iv) in the above Theorem 2.28 is also known as the *chain rule* of differentiation.

It is easily verified (see, e.g., Exercise 2.11) that the constant function is differentiable and its derivative is zero at any point. This also applies for the linear functions $f(x, y) = x$ and $f(x, y) = y$. Both are differentiable everywhere and its derivative is $[1 \ 0]$ and $[0 \ 1]$, respectively, at all points. By using the differentiability properties of scalar functions such as polynomials, rational functions, exponential and logarithmic functions, we can immediately conclude from Theorem 2.28 that sums, products and quotients of multi-variable functions of this type are also differentiable, provided the denominator is nonzero. Since the partial derivatives of these functions are themselves of the same type, it follows that these functions are typically differentiable as many times as needed. In other words, most of the functions we commonly work with can be differentiated repeatedly without difficulty.

EXAMPLE 2.29

1. The function $f(x, y) = x^4 - 2 + y$ is differentiable for all $(x_0, y_0) \in \mathbb{R}^2$. This follows directly from Theorem 2.28.

To see this, first note that the constant function -2 , $k_1(x, y) = x$ and $k_2(x, y) = y$ are differentiable everywhere. By Theorem 2.28, item ii), it follows that $f_1(x, y) = x^4$ is differentiable. By item i), this implies that $f(x, y)$ is differentiable.

At a fixed point (x_0, y_0) , the derivative of the section function $g(x) = f(x, y_0) = x^4 - 2 + y_0$ is $g'(x) = 4x^3$, so $D_1f(x_0, y_0) = 4x_0^3$. Furthermore, the derivative of the section function $h(y) = f(x_0, y) = x_0^4 - 2 + y$ is $h'(y) = 1$, so $D_2f(x_0, y_0) = 1$.

Consequently, the derivative of f at (x_0, y_0) is $f'(x_0, y_0) = [4x_0^3 \ 1]$. Using this, we can straightforwardly determine that the tangent plane to the graph of f at, for example, $(1, 2)$ is: $z(x, y) = 1 + 4(x - 1) + (y - 2)$.

2. The function $f(x, y) = \frac{xy}{1+x^2}$ is differentiable for all $(x_0, y_0) \in \mathbb{R}^2$. This follows directly from Theorem 2.28 again.

To see this, first note that the constant function 1 , $k_1(x, y) = x$ and $k_2(x, y) = y$ are differentiable everywhere. By Theorem 2.28, item ii), it follows that $f_1(x, y) = xy$ and $f_2(x, y) = x^2$ are differentiable. Then, by item i), $f_3(x, y) = 1 + x^2$ is differentiable everywhere. Since $f_3(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$, item iii) implies that $f(x, y) = \frac{f_1(x, y)}{f_3(x, y)}$ is differentiable everywhere.

At a fixed point (x_0, y_0) , the derivative of the section function $g(x) = f(x, y_0) = \frac{xy_0}{1+x^2}$ is

$$g'(x) = \frac{y_0(1+x^2) - 2x^2y_0}{(1+x^2)^2}.$$

So, $D_1f(x_0, y_0) = \frac{y_0(1+x_0^2) - 2x_0^2y_0}{(1+x_0^2)^2}$.

Similarly, the derivative of the section function $h(y) = f(x_0, y) = \frac{x_0y}{1+x_0^2}$ is $h'(y) = \frac{x_0}{1+x_0^2}$. Hence, $D_2f(x_0, y_0) = \frac{x_0}{1+x_0^2}$.

In particular, the derivative of f at $(0, 1)$ is $f'(0, 1) = [1 \ 0]$.

Therefore, the tangent plane to the graph of f at $(0, 1)$ is $z(x, y) = x$.

3. The function $f(x, y) = e^{x^2+y} + \ln(x^2 + y^2 + 1)$ is differentiable for all $(x_0, y_0) \in \mathbb{R}^2$.

This again follows directly from Theorem 2.28. To see this, first note that the functions $k_1(x, y) = x$ and $k_2(x, y) = y$ are differentiable everywhere. Therefore, by Theorem 2.28, item ii), $f_1(x, y) = x^2$ is also differentiable everywhere. By item i), it then follows that $f_2(x, y) = x^2 + y$ is differentiable everywhere.

Next, note that $k(z) = e^z$ is differentiable everywhere. By item iv), it follows that $f_3(x, y) = e^{x^2+y}$ is differentiable everywhere. Similarly, $f_4(x, y) = \ln(x^2 + y^2 + 1)$ is differentiable everywhere. Therefore, the final conclusion is that $f(x, y) = f_3(x, y) + f_4(x, y)$ is differentiable everywhere as well.

At a fixed point (x_0, y_0) , the derivative of the section function $g(x) = f(x, y_0) = e^{x^2+y_0} + \ln(x^2 +$

$y_0^2 + 1$), is

$$g'(x) = 2xe^{x^2+y_0} + \frac{2x}{x^2+y_0^2+1}.$$

So $D_1f(x_0, y_0) = 2x_0e^{x_0^2+y_0} + \frac{2x_0}{x_0^2+y_0^2+1}$.

Similarly, the derivative of the section function, $h(y) = f(x_0, y) = e^{x_0^2+y} + \ln(x_0^2 + y^2 + 1)$, is

$$h'(y) = e^{x_0^2+y} + \frac{2y}{x_0^2 + y^2 + 1}.$$

So $D_2f(x_0, y_0) = e^{x_0^2+y_0} + \frac{2y_0}{x_0^2+y_0^2+1}$.

Thus, for example, at $(0, 0)$ the derivative of f is $f'(0, 0) = [0 \ 1]$.

The tangent plane to the graph of f at $(0, 0)$ is therefore $z(x, y) = 1 + y$.

EXERCISE 2.12 Motivate why the following functions f are differentiable. Furthermore, determine their derivatives and the tangent plane to the graph of f at the indicated point.

- $f(x, y) = x^\alpha y^{1-\alpha}$, at $(x, y) = (1, 1)$, where α is some scalar in the interval $(0, 1)$.
- $f(x, y) = \frac{1}{2}x^2e^{y^2}$, at $(x, y) = (0, 0)$.
- $f(x, y) = \frac{x-y}{x+y}$, at $(x, y) = (1, 1)$.
- $f(x, y) = (\alpha x^2 + \beta y)^4$ at $(x, y) = (0, 1)$, where α, β are some positive numbers.
- $f(x, y, z) = \sqrt{xy} + \frac{z}{x}$ at $(x, y, z) = (1, 1, 0)$.

2.5 First-Order Condition for an Optimum of a Multi-Variable Function

To determine the optimum of a multi-variable objective function $f(x)$, $x \in C \subset \mathbb{R}^n$, we can proceed in a manner analogous to Algorithm 1.26 for the scalar case.

ALGORITHM 2.30 To find the global optima of a function, basically, proceed as follows.

Step 1. Locate all local maxima and minima of the objective function.

Step 2. Compare the function value at the local maxima and minima and at the boundaries of the domain. The highest (lowest) value yields then the global maximum (minimum) of the objective function.

To carry out Step 1, we first need to define what is meant by a local maximum or minimum. The idea is analogous to the scalar case. A function f is said to have a local maximum at the point $(x, y) = (x_0, y_0)$ if $f(x_0, y_0) \geq f(x, y)$, for all (x, y) within some ball around (x_0, y_0) . Similarly, f has a local minimum at $(x, y) = (x_0, y_0)$ if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) within some ball around (x_0, y_0) . Figure 2.15 illustrates the concept of a local minimum.

DEFINITION 2.31 Let f be defined on an open set $S \subset \mathbb{R}^n$ and let $s_0 \in S$. Then, we say that f has a *local maximum (minimum)* at s_0 if $f(s) \leq (\geq) f(s_0)$ for all values of s in some ball B with center s_0 . We say that f has a *global maximum (minimum)* at s_0 if $f(s) \leq (\geq) f(s_0)$ for all $s \in S$.

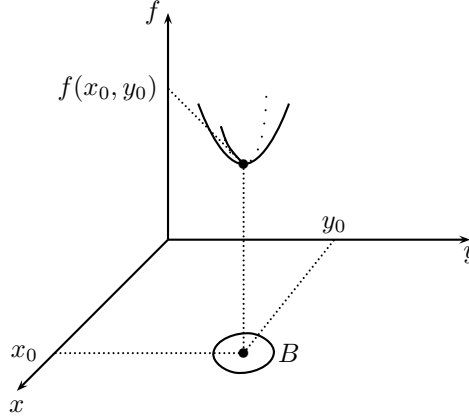


FIGURE 2.15 – Local minimum of f at (x_0, y_0) .

If the function $f(x, y)$ has an extremal point at (x_0, y_0) , then the section function $g(x) = f(x, y_0)$ must also have an extremum at $x = x_0$. Similarly, the section function $h(y) = f(x_0, y)$ must have an extremum at $y = y_0$. In other words, if f is differentiable, both partial derivatives of f must be zero at (x_0, y_0) . Geometrically, this means that the tangent plane to the graph of f at (x_0, y_0) is horizontal.

THEOREM 2.32 (*first-order condition*)

Suppose $f(s)$ is differentiable on an open set $S \subset \mathbb{R}^n$ and that $a \in S$. If f has a maximum or a minimum at $s = a$, $f'(a) = [0 \ \cdots \ 0]$.

This property will be used in the following to identify an initial set of candidate points where a function f may attain its optimum. As in the scalar case, points where the derivative of f is zero are called *stationary points*.

Like in the scalar case, stationary points of a function are not always optima. A stationary point s_0 , where the function in some directions has a minimum at s_0 and at other directions a maximum at s_0 , is called a *saddle point*. So, by definition, at a stationary point the function either has a local maximum, a local minimum or a saddle point.

EXAMPLE 2.33

1. Consider $f(x, y) = x^2 - 2x + y^2 - 4y + 5$. The stationary points of f are obtained by solving $f'(x, y) = [0 \ 0]$. That is they are obtained as the solution to the set of equations: $2x - 2 = 0$ and $2y - 4 = 0$. Clearly, this set of equations has only one solution, $(x, y) = (1, 2)$. So, f has just one stationary point in this case.

2. Consider $f(x, y) = xy^2 + x^3y - xy$. The stationary points of f are obtained by solving $f'(x, y) = [0 \ 0]$, i.e., $y^2 + 3x^2y - y = 0$ and $2xy + x^3 - x = 0$. These equations can be rewritten as follows.

$$y(y + 3x^2 - 1) = 0 \text{ and } x(2y + x^2 - 1) = 0.$$

From the first equation we have that either: i) $y = 0$ or ii) $y + 3x^2 - 1 = 0$. From the second equation we have that either: a) $x = 0$ or b) $2y + x^2 - 1 = 0$. So, all stationary points are obtained by determining the solutions of the set of equations I. i) and a); II. i) and b); III. ii) and a) and IV. ii) and b).

The set I. has the solution $(0, 0)$. The set II. has the solutions $(1, 0)$ and $(-1, 0)$. The set III. has the solution $(0, 1)$ and, finally, the set IV. has the solutions $(\sqrt{\frac{1}{5}}, \frac{2}{5})$ and $(-\sqrt{\frac{1}{5}}, \frac{2}{5})$.

So, f has 6 stationary points.

3. Consider $f(x, y, z) = -x^3 + 6x + 3xz + y^3 + 3yz$. The stationary points of f are found by solving $f'(x, y, z) = [0 \ 0 \ 0]$. This yields the next three equations

$$-3x^2 + 6 + 3z = 0; \quad 3y^2 + 3z = 0 \text{ and } 3x + 3y = 0.$$

From the last equation we have $y = -x$. Substitution of this into the second equation yields $z = -x^2$. From the first equation we get then that $x = 1$ or $x = -1$. So f has two stationary points, i.e., $(1, -1, -1)$ and $(-1, 1, -1)$.

EXAMPLE 2.34 (marginal productivity theory)

Consider a producer who aims to maximize his profit. Suppose his production function is given by $Q = f(K, L)$, where K and L denote the input factors capital and labor, respectively. Let ω denote the cost of one unit of labor and r the cost of one unit of capital. Assume the producer can not affect the market price p of the product. Then the profit, π , from producing and selling Q units equals

$$\pi(K, L) = pf(K, L) - \omega L - rK.$$

Assuming f is differentiable, the first-order conditions for (K_0, L_0) to be a maximum are

$$\pi'_K(K, L) = pf'_K - r = 0 \text{ and } \pi'_L(K, L) = pf'_L - \omega = 0.$$

Or, stated differently,

$$f'_K = \frac{r}{p} \text{ and } f'_L = \frac{\omega}{p}.$$

That is, at an optimum, both marginal productivity of capital and marginal productivity of labor should equal their relative prices, i.e., $\frac{r}{p}$ and $\frac{\omega}{p}$, respectively.

EXERCISE 2.13 Determine the stationary points of the functions defined by the following formulas:

- | | | |
|---------------------------|---------------------------------|---|
| a. $x^2 - y^2$. | c. $x^3 - y^3 + 9xy$. | e. $(x - y)^2 + (z - y)^2 + y^4 - 2y^2 + 1$. |
| b. $\ln(1 + x^2 + y^2)$. | d. $\frac{x^2 + 2x}{1 + y^2}$. | f. $y^2 e^{x^2}$. |

2.6 Convexity and Second-Order Condition for an Optimum of Multi-Variable Functions

In this section, we examine the behavior of multi-variable functions near their optimal points in greater detail. This analysis will ultimately lead to a generalization of the second-order optimality criteria introduced in Chapter 1 for scalar functions. To illustrate this, consider the local minimum depicted in Figure 2.15.

Roughly speaking, f has a local minimum at (x_0, y_0) since every section function has a minimum at (x_0, y_0) . Recall from Chapter 1 that a scalar function attains a minimum at a stationary point if the function is convex on an open interval containing that point. In other words, for any chord P_1Q_1 with endpoints P_1 and Q_1 on the graph of f , every point on the line segment connecting them lies on or above the graph of f . This geometric property can be expressed formally by the following inequality, which holds for all x, y and $\bar{\lambda} \in [0, 1]$:

$$f(\bar{\lambda}x + (1 - \bar{\lambda})y) \leq \bar{\lambda}f(x) + (1 - \bar{\lambda})f(y).$$

From the graph of f in Figure 2.15 we can see that this property also characterizes the "isolated" minimum shown. Therefore, we first generalize the notions of convexity/concavity for multi-variable functions. We will call a function $f : S \rightarrow \mathbb{R}$ *convex* (concave) if for every two points $x, y \in D$ the intersection curve corresponding with the line segment between x and y is convex (concave).

DEFINITION 2.35 Let f be defined on an open *convex set*⁷ $S \subset \mathbb{R}^n$. Then, f is called

– *convex* on S if it is true that, for each $\lambda \in [0, 1]$

$$f(\lambda s_1 + (1 - \lambda)s_2) \leq \lambda f(s_1) + (1 - \lambda)f(s_2),$$

whenever $s_1 \in S$ and $s_2 \in S$.

– *concave* on S if it is true that, for each $\lambda \in [0, 1]$

$$f(\lambda s_1 + (1 - \lambda)s_2) \geq \lambda f(s_1) + (1 - \lambda)f(s_2),$$

whenever $s_1 \in S$ and $s_2 \in S$.

⁷ A set S is called *convex* if, with any two points in S also the line connecting them belongs to S . This technical assumption is needed to make sure that the function is well-defined at any interior point of S .

If a strict inequality holds for all $\lambda \in (0, 1)$, and $s_1 \neq s_2$, we say that f is *strictly convex* (concave, respectively).

Once again, it follows directly from the definitions that the following relationship holds between convex and concave functions.

PROPOSITION 2.36 *Let f be defined on an open convex set S . Then, f is concave on S if and only if $-f$ is convex on S .*

Like for the scalar case it can be shown that every convex (concave) function is always continuous.

Before we present a characterization of a convex function f by means of its second-order derivative we first consider the analogue of the scalar case first-order characterization. Figure 2.16 illustrates this first-order characterization.

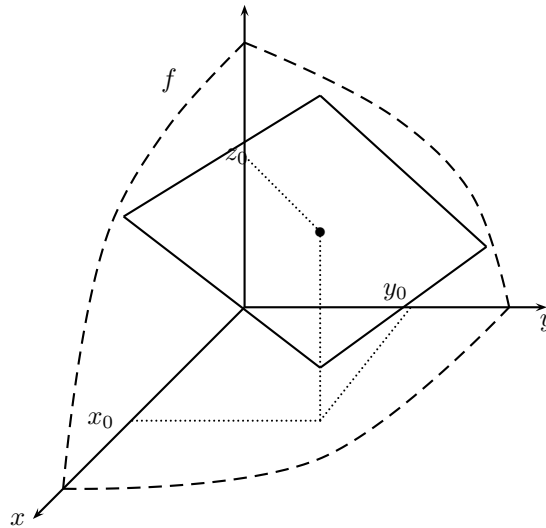


FIGURE 2.16 – Tangent plane at (x_0, y_0) to concave curve

In this graph we plotted a concave function $f(x, y)$. We observe that, at every point on the graph, the function f lies entirely below the tangent plane at that point. Theorem 2.37 formalizes this observation.

THEOREM 2.37 (First-Order Characterization convex/concave function) *Let f be differentiable on an open convex set S . Then, f is*

- convex on S if and only if $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in S$;
- concave on S if and only if $f(y) \leq f(x) + f'(x)(y - x)$ for all $x, y \in S$.

Now suppose that the conditions of Theorem 2.37 are satisfied. If f has a stationary point at $x = a$, then $f(y) \geq f(a)$ for all $y \in S$ when f is convex (and $f(y) \leq f(a)$ when f is concave). In other words, in this case f attains a global minimum (respectively, a global maximum) at $x = a$. Corollary 2.38 formalizes this result.

COROLLARY 2.38 *(Stationary point convex/concave function gives global extremum) Let f be differentiable on an open convex set S , and suppose $Df(a) = 0$. Then f attains a global minimum at $x = a$ if f is convex on S , and a global maximum at $x = a$ if f is concave on S .*

A similar remark to the one made in the scalar case regarding the practical relevance of Theorem 2.37 also applies here. Specifically, this result can be used to approximate the graph of an unknown convex function f , by evaluating its value and derivative at a number of points.

EXERCISE 2.14

- Consider $f(x, y) = -x^2 - y^2$. Determine the tangent plane to the graph of $f(x, y)$ at (a, b) .
- Show that $f(x, y) \leq f(a, b) + Df(a, b)[x - a \ y - b]^T$ for all $(x, y) \in \mathbb{R}^2$.
- Verify that $f(x, y)$ is concave on $\mathbb{R}^2$.

Now, if a scalar function f is twice differentiable on an open interval I , then f is convex (concave) on I if and only if its second-order derivative is nonnegative (nonpositive) on I . Hence, a multi-variable function is convex (respectively concave) in a neighborhood of a point (x_0, y_0) if and only if all intersection curves of the graph of f with vertical planes passing through (x_0, y_0) are convex (respectively concave). In other words, according to the second-order characterization, each of these intersection curves has a nonnegative (respectively, nonpositive) second-order derivative. In particular, this implies that the second-order derivative of the section function $g(x) = f(x, y_0)$, which is $\frac{\partial^2 f}{\partial x^2}$, and the second-order derivative of the section function $h(y) = f(x_0, y)$, which is $\frac{\partial^2 f}{\partial y^2}$, must both be nonnegative (nonpositive). Clearly, since these are only two intersection curves among infinitely many that must be examined, it remains an open question whether these two nonnegativity (respectively, nonpositivity) conditions are sufficient to conclude that a multi-variable function is convex (respectively, concave). To answer this question recall that by an appropriate choice of parameters α and β we can represent any straight line in the plane through some point (x_0, y_0) as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \text{ where } t \in (-\infty, \infty).$$

For instance, by choosing $\alpha = 1$ and $\beta = 0$ we get the horizontal line through the point (x_0, y_0) . Figure 2.17 illustrates the idea.

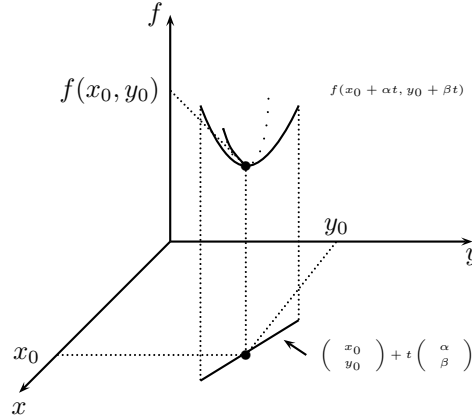


FIGURE 2.17 – Arbitrary intersection curve of f at (x_0, y_0) .

Using this representation, any intersection curve of the graph corresponding to a line through the point (x_0, y_0) can be expressed as

$$F(t) := f(x_0 + \alpha t, y_0 + \beta t), \text{ where } t \in (-\infty, \infty).$$

So, $f(x, y)$ is convex if for every choice of α and β the function $F(t)$ is convex.

Since $F(t)$ is a scalar function, Theorem 2.37 tells us that (assuming f is twice differentiable) $F(t)$ is convex if and only if $F''(t) \geq 0$. Applying the standard rules of differentiation, we obtain

$$F'(t) = \alpha f'_x(x_0 + \alpha t, y_0 + \beta t) + \beta f'_y(x_0 + \alpha t, y_0 + \beta t).$$

Differentiating this expression once more and reorganizing terms gives the following expression for $F''(t)$:

$$\begin{aligned} \alpha^2 f''_{xx}(x_0 + \alpha t, y_0 + \beta t) + \alpha\beta f''_{xy}(x_0 + \alpha t, y_0 + \beta t) + \\ \alpha\beta f''_{yx}(x_0 + \alpha t, y_0 + \beta t) + \beta^2 f''_{yy}(x_0 + \alpha t, y_0 + \beta t). \end{aligned}$$

Which can be rewritten as the quadratic form:

$$F''(t) = [\alpha \ \beta] \begin{bmatrix} f''_{xx}(x, y) & f''_{xy}(x, y) \\ f''_{yx}(x, y) & f''_{yy}(x, y) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where we used the notation f''_{xy} to denote the partial derivative of f'_x w.r.t. y , etc.

From this we conclude that $f(x, y)$ will be convex if and only if for every choice of α and β

$$[\alpha \ \beta] \begin{bmatrix} f''_{xx}(x, y) & f''_{xy}(x, y) \\ f''_{yx}(x, y) & f''_{yy}(x, y) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \geq 0, \text{ for all } (x, y).$$

Before we proceed with a more detailed analysis of this result, we first introduce some new definitions and intermediate results.

As noted earlier, the partial derivative of a multi-variable function f is itself a multi-variable function, so we can take partial derivatives of it as well. These are called the second-order partial derivatives of f .

DEFINITION 2.39 Let f be defined on an open set S . Assume, f is partial differentiable in a ball around $s \in S$.

If $D_j f$ is partial differentiable in s with respect to the i -th variable s_i , then $D_i(D_j f)(s)$ is called the *second-order partial derivative* of f in s with respect to the i -th and j -th variable and denoted by $D_i(D_j f)(s) = D_{ij} f(s)$ (or $\frac{\partial^2 f}{\partial s_i \partial s_j}(s)$).

f is called *twice partial differentiable* in s if all second-order partial derivatives in s exist. If f is twice partial differentiable in every point of S , f is called *twice partial differentiable* on S .

In case a function is twice partial differentiable on S , and all second-order partial derivatives are continuous on S , f is called *twice continuously differentiable*. Notation: $f \in C^2$.

EXAMPLE 2.40

The function $f(x, y) = x^3 + y^4$ is C^2 on $\mathbb{R}^2$ as: $D_1 f(x, y) = 3x^2$, $D_2 f(x, y) = 4y^3$, $D_{11} f(x, y) = 6x$, $D_{12} f(x, y) = D_{21} f(x, y) = 0$, $D_{22} f(x, y) = 6y$. And, clearly, $D_{ij} f$ are continuous on $\mathbb{R}^2$.

EXERCISE 2.15 Calculate all second-order partial derivatives of next functions f .

- a. $f(x, y) = xy^2$. b. $f(x, y) = e^{xy} + x^3$. c. $f(x, y) = \frac{x}{y}$, $y \neq 0$. d. $f(x, y, z) = xy + yz$.

An important property of twice continuously differentiable functions is that the order of differentiation does not matter. In other words, whether we first differentiate with respect to s_i and then with respect to s_j , or vice versa, the result is the same.

THEOREM 2.41 Let $f \in C^2$ in a ball around s_0 . Then, $D_{ij} f(s_0) = D_{ji} f(s_0)$, for every $i, j \in \{1, \dots, n\}$.

EXAMPLE 2.42 Consider $f(x, y) = xy^2 + \ln(x^2y + 1)$, for $x > 0$, $y > 0$. Then,

$$D_1 f(x, y) = y^2 + \frac{2xy}{x^2y + 1} \text{ and } D_2 f(x, y) = 2xy + \frac{x^2}{x^2y + 1}.$$

From the first equation it follows then

$$D_{11} f(x, y) = \frac{2y(x^2y + 1) - 4x^2y^2}{(x^2y + 1)^2} \text{ and } D_{12} f(x, y) = 2y + \frac{2x(x^2y + 1) - 2x^3y}{(x^2y + 1)^2}.$$

Whereas, from the second equation we get

$$D_{21}f(x, y) = 2y + \frac{2x(x^2y + 1) - 2x^3y}{(x^2y + 1)^2} \text{ and } D_{22}f(x, y) = 2x - \frac{x^4}{(x^2y + 1)^2}.$$

Clearly, all second-order partial derivatives are continuous. So, $f \in C^2$.

A simple inspection of $D_{12}f$ and $D_{21}f$ shows that both indeed coincide.

Similar to the first-order partial derivatives, all partial derivatives of $D_i f(s)$ can be organized in a single row, i.e.,

$$[D_{i1}f(s) \ D_{i2}f(s) \ \cdots \ D_{in}f(s)].$$

By stacking all these rows into a single matrix, we then obtain the following matrix of second-order partial derivatives:

$$Hf(s) = \begin{bmatrix} D_{11}f(s) & \cdots & D_{1n}f(s) \\ \vdots & & \vdots \\ D_{n1}f(s) & \cdots & D_{nn}f(s) \end{bmatrix}.$$

This matrix $Hf(s)$ is called the *Hessian* of f at s , or the second-order derivative of f at s , f'' . From Theorem 2.41 it follows now directly that, if $f \in C^2$, the Hessian $Hf(s)$ is a symmetric matrix (i.e., $Hf(s)$ equals its transpose).

EXAMPLE 2.43 Consider $f(x, y, z) = xyz + x^2$. Then, $f'(x, y, z) = [yz + 2x \ xz \ xy]$.

$$\text{So, } Hf(x, y, z) = \begin{bmatrix} 2 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix} \text{ and, therefore, } Hf(1, 2, 3) = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

EXERCISE 2.16 Calculate Hf for the next functions f .

a. $f(x, y) = x^\alpha y^{1-\alpha}$.

b. $f(x, y, z) = (x^2y - y^2x)z$.

Next, for a given $n \times n$ symmetric matrix A , consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$f(s) = s^T A s, \text{ where } s^T \text{ denotes the transpose of } s. \quad (2.3)$$

One can easily verify that $f(s)$ is a *quadratic form*. In fact, it is also easy to see that any quadratic form in $\mathbb{R}^n$ can be rewritten as (2.3).

EXAMPLE 2.44

1. Straightforward calculations show that $f(x, y) = [x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$.
2. Consider $f(x, y, z) = x^2 + 4xy + 2xz - 6yz - y^2 + 5z^2$. Then,

$$f(x, y, z) = [x \ y \ z] \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

EXERCISE 2.17 Rewrite the quadratic forms, below, as $s^T A s$, with A a symmetric matrix.

- a. $f(x, y) = 2x^2 + 2xy + 3y^2$. b. $f(x, y, z) = xy - 2yz + 4xz$. c. $f(x, y, z) = 2x^2 + 2yz$.

DEFINITION 2.45 The quadratic form (2.3) (or, matrix A) is called

- *positive definite*, if for every $s \in \mathbb{R}^n$, $s \neq 0$, we have: $f(s) > 0$.
- *positive semi-definite*, if for every $s \in \mathbb{R}^n$, $s \neq 0$, we have: $f(s) \geq 0$.
- *negative definite*, if for every $s \in \mathbb{R}^n$, $s \neq 0$, we have: $f(s) < 0$.
- *negative semi-definite*, if for every $s \in \mathbb{R}^n$, $s \neq 0$, we have: $f(s) \leq 0$.
- *indefinite*, if there exist $s_i \in \mathbb{R}^n$, $i = 1, 2$, with $f(s_1) > 0$ and $f(s_2) < 0$.

We will also use the notation $A > 0$, $A \geq 0$, $A < 0$ and $A \leq 0$ in case the quadratic form (2.3) is positive definite, positive semi-definite, negative definite and negative semi-definite, respectively. Note that this notation does **not** mean that every entry of the matrix is subject to the specified sign constraints. For instance, the quadratic form $x^2 + (x - y)^2 + y^2$ is clearly positive definite, whereas not all entries of the corresponding matrix in the quadratic form, $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, are positive.

Note that if a quadratic form is positive definite it is also positive semi-definite. Furthermore, it follows directly from the definition that a quadratic form f is positive (semi-) definite if and only if $-f$ is negative (semi-) definite.

We can now restate our previous observation regarding when a twice continuously differentiable multi-variable function is concave (respectively, convex) as follows:

THEOREM 2.46 (*second-order Characterization convex/concave function*) Let $S \subset \mathbb{R}^n$ be an open convex set and suppose f is twice continuously differentiable on S . Then, f is

- *convex on S* if and only if $f''(s)$ is positive semi-definite for all $s \in S$;
- *concave on S* if and only if $f''(s)$ is negative semi-definite for all $s \in S$.

Using the above characterization, one can easily show that if two convex (concave) functions that are twice continuously differentiable, their sum will be convex (concave) too.

An obvious question that arises next is how we can easily verify whether a quadratic form has either one of these definiteness properties. There are two, well-known, criteria to check definiteness of a symmetric matrix A . The first one is by checking its, so-called, eigenvalues. We will get back to this

later on. The other criterion involves checking whether certain determinants derived from this matrix A satisfy specific sign conditions.

Recall that if A is a square $n \times n$ matrix with entries a_{ij} its determinant, $\det A$, is defined as follows: for $n = 1$, it is simply the scalar A ; for larger matrices, it is recursively computed from A as:

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{n+1}a_{1n}\det A_{1n},$$

where A_{ij} denotes the submatrix of A formed by deleting row i and column j .

EXAMPLE 2.47

1. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then the determinant of matrix A is the scalar: $\det A = a_{11}a_{22} - a_{21}a_{12}$.
2. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$. Then the determinant of matrix A is the scalar:

$$\begin{aligned} \det A &= 0 * \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} - 1 * \det \begin{bmatrix} 3 & 5 \\ 6 & 8 \end{bmatrix} + 2 * \det \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} \\ &= 0 * (32 - 35) - 1 * (24 - 30) + 2 * (21 - 24) = 0. \end{aligned}$$

A square matrix which determinant is zero is called *singular*. A square matrix which has a determinant different from zero is called *invertible*. Recall that a set of square linear equations $Ax = b$ has a unique solution if and only if the determinant of A differs from zero. Readers who want to refresh their knowledge on this are referred to any textbook on linear algebra (see, e.g., [13]).

DEFINITION 2.48 Let A be an $n \times n$ symmetric matrix.

The $k \times k$ *leading principal submatrix* A_k of A is the matrix, which is composed of the elements in the first k rows and columns of A . The determinant of this matrix is called a *leading principal minor*.

A $k \times k$ *principal submatrix* of A is a matrix, which is composed of the elements in k (arbitrary) rows and corresponding columns of A . The determinant of this matrix is called a *principal minor*.

EXAMPLE 2.49 Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & -3 & 5 \end{bmatrix}$. The leading principal submatrices of A are: $[1]$; $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and A . The leading principal minors of A are, consequently, $\det[1] = 1$, $\det \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = -5$ and $\det A = -45$.

The principal matrices of A are the matrices

$$[1]; [-1]; [5]; \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}; \begin{bmatrix} -1 & -3 \\ -3 & 5 \end{bmatrix} \text{ and } A.$$

Therefore, the principal minors are: 1, -1, 5, -5, -2, 4 and -45.

EXERCISE 2.18 Calculate all principal matrices and principal minors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

To determine whether a symmetric matrix A is positive (semi-)definite or negative (semi-)definite, one can use the following characterization, which is based on checking the signs of the (leading) principal minors of the matrix.

PROPOSITION 2.50 *Let A be an $n \times n$ symmetric matrix. Then,*

- A is positive definite if and only if all leading principal minors are larger than zero.
- A is positive semi-definite if and only if all principal minors are nonnegative.
- A is negative definite if and only if $-A$ is positive definite.
- A is negative semi-definite if and only if $-A$ is positive semi-definite.

REMARK 2.51 Another, sometimes more convenient, way to check whether a matrix A is positive semi-definite is based on the following property: a matrix A is positive semi-definite if and only if, for every positive scalar ϵ , the matrix $A + \epsilon I$ is positive definite.

Therefore, by verifying that all leading principal minors of $A + \epsilon I$ are positive for every $\epsilon > 0$, one can also determine whether A is positive semi-definite.

EXAMPLE 2.52

1. Consider $A = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$. Then $\det[2] > 0$, $\det[4] > 0$ and $\det A = 4 > 0$. So, A is positive definite.

2. Consider $A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$. Then $\det[-1] < 0$. So, A is **neither** positive definite or positive semi-

definite. To see whether A is negative (semi-) definite, we consider $B = -A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$. Note that $\det[1] > 0$, $\det[4] > 0$ and $\det B = 0$. So, B is positive semi-definite, but **not** positive definite. Therefore, A is negative semi-definite but **not** negative definite.

This conclusion results also immediately if we follow the suggestion made in Remark 2.51. That is, verify whether matrix $-A + \epsilon I = \begin{bmatrix} 1 + \epsilon & -2 \\ -2 & 4 + \epsilon \end{bmatrix}$ is positive definite for every positive ϵ . It is readily verified that $\det[1 + \epsilon] > 0$, for every positive ϵ , and $\det[-A + \epsilon I] = 5\epsilon + \epsilon^2 > 0$, for

every positive ϵ too. So, for every positive ϵ , $-A + \epsilon I$ is positive definite. Therefore, $-A$ is positive semi-definite and, thus, A is negative semi-definite.

3. Consider $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$. Then $\det[1] > 0$, $\det[2] > 0$ and $\det A = -7 < 0$. So, A is not positive semi-definite. Furthermore, $B = -A = \begin{bmatrix} -1 & -3 \\ -3 & -2 \end{bmatrix}$. Since $\det[-1] < 0$, B is neither positive semi-definite. So, A is neither negative semi-definite. Consequently, since A is neither positive semi-definite or negative semi-definite, A is indefinite.

This could also have been verified directly from matrix A . For, e.g., $[1 \ 0]A[1 \ 0]^T = 1 > 0$. Whereas, e.g., $[1 \ -1]A[1 \ -1]^T = -3 < 0$.

EXAMPLE 2.53

1. Consider $f(x, y, z) = x^2 + 2xy + 2y^2 + z^4$. To see whether $f(x, y, z)$ is convex, we calculate its second-order derivative. Straightforward calculations show that $f'(x, y, z) = [2x + 2y \ 2x + 4y \ 4z^3]$ and, consequently,

$$f''(x, y) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 12z^2 \end{bmatrix}.$$

The principal minors of f'' are $\det[2] \geq 0$, $\det[4] \geq 0$, $\det[12z^2] \geq 0$, $\det \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = 4 \geq 0$,

$$\det \begin{bmatrix} 2 & 0 \\ 0 & 12z^2 \end{bmatrix} = 24z^2 \geq 0, \det \begin{bmatrix} 4 & 0 \\ 0 & 12z^2 \end{bmatrix} = 48z^2 \geq 0, \text{ and } \det f'' = 48z^2 \geq 0.$$

Therefore, by Theorem 2.46, f is convex.

2. Consider $f(x, y) = x^4 + 2x^2y^2 + y^2$. To see whether $f(x, y)$ is convex, we calculate its second-order derivative. Straightforward calculations show that $f' = [4x^3 + 4xy^2 \ 4x^2y + 2y]$ and, consequently,

$$f''(x, y) = \begin{bmatrix} 12x^2 + 4y^2 & 8xy \\ 8xy & 4x^2 + 2 \end{bmatrix}.$$

The principal minors of f'' are $\det[12x^2 + 4y^2] \geq 0$, $\det[4x^2 + 2] \geq 0$ and $\det f'' = (12x^2 + 4y^2)(4x^2 + 2) - 64x^2y^2 = 48x^2(x^2 - y^2) + 24x^2 + 8y^2$. Note that, e.g., $\det f''(1, 2) < 0$. So, $f(x, y)$ is not convex.

3. Consider the Cobb-Douglas production function $Q(K, L) = K^\alpha L^{1-\alpha}$, $\alpha \in (0, 1)$, for positive values of K and L . Then, $Q' = [\alpha K^{\alpha-1} L^{1-\alpha} \ (1-\alpha)K^\alpha L^{-\alpha}]$ and, consequently,

$$Q''(K, L) = \begin{bmatrix} \alpha(\alpha-1)K^{\alpha-2}L^{1-\alpha} & \alpha(1-\alpha)K^{\alpha-1}L^{-\alpha} \\ \alpha(1-\alpha)K^{\alpha-1}L^{-\alpha} & -\alpha(1-\alpha)K^\alpha L^{-\alpha-1} \end{bmatrix}.$$

The principal minors of $-Q''(K, L)$ are: $\alpha(1 - \alpha) \det[K^{\alpha-2} L^{1-\alpha}] \geq 0$, $\alpha(1 - \alpha) \det[K^\alpha L^{-\alpha-1}] \geq 0$ and $\det Q''(K, L) = \alpha^2(1 - \alpha)^2 K^{\alpha-2} L^{1-\alpha} K^\alpha L^{-\alpha-1} - \alpha^2(1 - \alpha)^2 [K^{\alpha-1} L^{-\alpha}]^2 = 0$.

So, the Cobb-Douglas production function is concave on its domain.

EXERCISE 2.19 Verify whether the next functions f are either convex, concave or neither of them on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

- a. $f(x, y) = x^2 + y^2$. b. $f(x, y) = x^2 - y^2$. c. $f(x, y, z) = -x^2 - y^2 - z$.

EXERCISE 2.20 Verify whether the next functions f are either convex, concave or neither of them on $S = \{(x, y) \mid x > 0, y > 0\}$.

- a. $f(x, y) = e^{xy}$. c. $f(x, y) = e^{-x^2 - y^2}$.
b. $f(x, y) = \ln(x + y + 1)$. d. $f(x, y) = (1 + x + y)^\alpha, \alpha \in (0, 1)$.

Note that affine functions are both convex and concave. In fact, these are the only functions that can have both properties simultaneously.

Next, we return to the main item of this section, i.e., to characterize maxima and minima of a multi-variable function. As we already indicated, a function will have a maximum (minimum) at a stationary point s_0 if the function is concave (convex) on a ball containing s_0 . So, using our previous characterization of concavity of functions, we get the next result.

THEOREM 2.54 *Let f be twice continuously differentiable on an open set S and $s_0 \in S$. Assume $f'(s_0) = 0$. Then, f has a local*

- *maximum at s_0 if $Hf(s) \leq 0$ on a ball B containing s_0 ;*
- *minimum at s_0 if $Hf(s) \geq 0$ on a ball B containing s_0 .*

Similar to the scalar case, one can derive from this directly a sufficient condition, that can be easily verified, to conclude whether a stationary point s_0 will be a maximum or minimum location.

THEOREM 2.55 *(sufficient second-order condition)*

Let f be twice continuously differentiable on an open set S containing s_0 . Assume that s_0 is a stationary point of f . Then $f(s_0)$ will be

- *a local maximum if $Hf(s_0) < 0$;*
- *a local minimum if $Hf(s_0) > 0$;*
- *a saddle point if $Hf(s_0)$ is indefinite.*

From Theorem 2.55 immediately follows next necessary condition for the existence of either a maximum or a minimum at a stationary point.

COROLLARY 2.56 (*necessary second-order condition*)

Let f be twice continuously differentiable on an open set S containing s_0 . Assume that s_0 is a stationary point of f . Then if $f(s_0)$ is a

- local maximum $Hf(s_0) \leq 0$;
- local minimum $Hf(s_0) \geq 0$.

In case $Hf(s_0)$ is either positive semi-definite or negative semi-definite, Theorem 2.55 does not provide a unequivocal conclusion. Indeed, in either of those two cases, no conclusions can be drawn about the optimality of the stationary point s_0 . In such cases the function can either attain a maximum, a minimum or a saddle point at the stationary point s_0 . Next Example 2.57 illustrates this point.

EXAMPLE 2.57

1. Consider $f(x, y) = x^4 + y^4$. Simple calculations show, f has a stationary point at $(0, 0)$, $Hf(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and f has a global minimum at $(0, 0)$ since $f(x, y) \geq 0$.
2. Consider $f(x, y) = -(x^4 + y^4)$. Again, it is easily verified that, f has a stationary point at $(0, 0)$, $Hf(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and f has a global maximum at $(0, 0)$ since $f(x, y) \leq 0$.
3. Consider $f(x, y) = x^3 + y^3$. Direct calculations show again that, f has a stationary point at $(0, 0)$ and $Hf(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. As, e.g., $f(x, 0) = x^3$ attains both negative and positive values in every ball around $(x, y) = (0, 0)$ it follows that f has a saddle point at $(0, 0)$.

EXAMPLE 2.58

Consider $f(x, y, z) = -x^2 - 4y^2 - 4z^3 + 6x + 8yz + 20z$. To find the optima of this function, we first calculate its stationary points. Note

$$f' = [-2x + 6 \quad -8y + 8z \quad -12z^2 + 8y + 20].$$

From $D_1f = 0$, we immediately have that $x = 3$. From $D_2f = 0$, we get $y = z$. Substitution of this into $D_3f = 0$ shows that z satisfies $-12z^2 + 8z + 20 = 0$. So, $z = -1$ or $z = \frac{5}{3}$. Therefore, f has two stationary points: $(3, -1, -1)$ and $(3, \frac{5}{3}, \frac{5}{3})$.

To see whether f has an optimum at these stationary points we consider f'' .

$$f''(x, y, z) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -8 & 8 \\ 0 & 8 & -24z \end{bmatrix}.$$

Consequently,

$$Hf(3, -1, -1) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -8 & 8 \\ 0 & 8 & 24 \end{bmatrix} \quad \text{and} \quad Hf(3, \frac{5}{3}, \frac{5}{3}) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -8 & 8 \\ 0 & 8 & -40 \end{bmatrix}.$$

It is easily verified that $Hf(3, -1, -1)$ is indefinite. So, f has a saddle point at $(3, -1, -1)$. Clearly, $Hf(3, \frac{5}{3}, \frac{5}{3})$ is not positive definite as the first entry of this matrix is -2 . Next consider $B = -Hf(3, \frac{5}{3}, \frac{5}{3})$. It is straightforwardly verified that B is positive definite. Therefore, $Hf(3, \frac{5}{3}, \frac{5}{3})$ is negative definite. So, by Theorem 2.55, f has a local maximum in $(3, \frac{5}{3}, \frac{5}{3})$. Note that this maximum is not a global maximum since, e.g., $f(0, 0, 100) > f(3, \frac{5}{3}, \frac{5}{3})$.

EXERCISE 2.21 Determine all optima (location, value, kind (local/global)) of the functions

- a. $f(x, y) = xy^2 + x^3y - xy$. c. $f(x, y) = e^{xy}$.
b. $f(x, y, z) = (x - y)^2 + (z - y)^2 + y^4 - 2y^2 + 1$.

EXERCISE 2.22 Consider the set $D = \{(x, y) \mid x \geq 0, y \geq 0, x \leq 2, y \leq 2\}$. Determine all global optima (location, value, kind) of the next functions on D

- a. $f(x, y) = x - y + 1$. b. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$.

EXERCISE 2.23 Consider a monopolist who produces two products which are imperfect substitutes. The inverse demand q_i for product i , $i = 1, 2$, is given by $p_1 = 60 - q_1 - \frac{1}{2}q_2$ and $p_2 = 60 - q_2 - \frac{1}{2}q_1$, respectively. Here, p_i is the price for good i .

- Assume that the monopolist can produce both goods at zero marginal cost. Determine the optimal quantities the profit maximizing monopolist will produce. How high are his profits?
- Assume authorities in market 1 impose a tax of t per unit sold in market 1. Discuss the consequences.

Mixed Exercises

EXERCISE 2.24 Determine for the functions f below all stationary points. Indicate whether f attains at these points a local or global minimum/maximum.

- $f(x, y) = x^2 + 2x + y^2 - 4y.$
- $f(x, y, z) = (x - 1)(y - 1) + z^3.$
- $f(x, y, z) = -x^2 - 4y^2 - z^3 + 6x + 2xy + 12z.$
- $f(x, y) = \frac{1}{2}x^4 - 2x^2 + 2xy - y^2.$
- $f(x, y) = x^2 - 2xy + \frac{1}{2}y^4.$
- $f(x, y) = \frac{1}{2}x^4 - x^2 + y^2 - 2y + \frac{3}{2}.$
- $f(x, y) = x^3 + 3xy + \frac{3}{2}y^2.$
- $f(x, y) = x^2y + xy + x.$
- $f(x, y) = (x + y^2)e^{-\frac{1}{2}x^2}.$
- $f(x, y) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{2}y^2.$

EXERCISE 2.25 Determine for next functions f whether they are convex, respectively, concave.

- a. $f(x, y) = (x - y)^2$, $x > 0$, $y > 0$.
 b. $f(x, y) = x^3$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.
 c. $f(x, y) = e^{-xy}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.
 d. $f(x, y) = xy$, $x > 0$, $y > 0$.
 e. $f(x, y, z) = x^2 + y^2 - z^2$, $x > 0$, $y > 0$, $z < 0$.
 f. $f(x, y) = xy$, $x \in \mathbb{R}$, $y = 1$.

EXERCISE 2.26 Determine for next functions f the tangent plane to the graph of f at the indicated point.

- a. $f(x, y) = 2^{x+y}$ at $(x, y) = (1, 0)$.
 b. $f(x, y) = \frac{x+y}{x-y}$ at $(x, y) = (1, 0)$.
 c. $f(x, y, z) = x^2 + yz$ at $(x, y, z) = (1, 1, 1)$.
 d. $f(x, y, z) = x + 2y + 3z$ at $(x, y, z) = (1, 1, 1)$.

EXERCISE 2.27 Let $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ and $\mathbb{R}_+^3 := \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0\}$. Motivate which functions f , below, are differentiable on the indicated set D .

- a. $f(x, y) = \ln(1 - e^{-xy})$, $D = \mathbb{R}_+^2$.
 b. $f(x, y) = e^{x+\ln(y)}$, $D = \{(x, y) \mid x > -1, y > -1\}$.
 c. $f(x, y) = \frac{2^{xy} - xy}{2^{-xy} + xy}$, $D = \mathbb{R}_+^2$.
 d. $f(x, y) = \frac{\sqrt{x+y-2}}{x^2+y^2+1}$, $D = \mathbb{R}_+^2$.
 e. $f(x, y) = |x + y + 1|$, $D = \mathbb{R}_+^2$.
 f. $f(x, y, z) = \frac{\ln(1+e^{-xyz})}{e^{-xyz}+1}$, $D = \mathbb{R}_+^3$.

EXERCISE 2.28 Motivate which of next functions f has a global maximum on its domain D .

- a. $f(x, y) = -x^2 - y^2$, $D = \{(x, y) \mid x \geq 0, y > 0\}$.
 b. $f(x, y) = \frac{e^{xy} + 2^{xy}}{x+y+4}$, $D = \{(x, y) \mid |x| \leq 1, |y| \leq 2\}$.
 c. $f(x, y) = \frac{\ln(1+x^2+y^2)}{e^{1+x^2+y^2}}$, $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$.
 d. $f(x, y) = x^2 + \sqrt{y^2 - 4}$, $D = \{(x, y) \mid -4 \leq x \leq -2 \text{ and } -4 \leq y \leq -2; \text{ or } 2 \leq x \leq 4 \text{ and } 2 \leq y \leq 4\}$.
 e. $f(x, y) = \begin{cases} \frac{1}{x+y}, & x+y \neq 0; \\ 0, & x+y = 0; \end{cases}$ $D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$.
 f. $f(x, y) = \frac{1}{x+y+1}$, $D = \{(x, y, z) \mid x \geq 0, y \geq 0\}$.

EXERCISE 2.29 Determine the boundary and accumulation points of next sets S . Also motivate whether they are open/closed/bounded/compact, respectively.

- a. $\{(x, y) \mid 1 < x \leq 2, y \in \mathbb{R}; \text{ or } 4 \leq x \leq 5, 0 \leq y \leq 1\}$.
 b. $\{(x, y) \mid y \leq x \leq y+1 \text{ and } 0 \leq y+x \leq 1\}$.
 c. $\{(x, y) \mid x > 0 \text{ and } y > 0\}$.
 d. $\{(x, y) \mid x \leq 0 \text{ and } y > 0\}$.

EXERCISE 2.30 Motivate which functions f , below, are continuous on the indicated set S .

- a. $f(x, y) = \frac{\ln(x+y+1)}{x+y+1}$, $S = \{(x, y) \mid x > 0, y > 0\}$.
 b. $f(x, y) = |x| + |y|$, $S = \mathbb{R}^2$.
 c. $f(x, y) = \begin{cases} x+y-1, & \text{if } x > 0; \\ -x^2+xy+y-1, & \text{if } x \leq 0. \end{cases}$
 d. $f(x, y) = \begin{cases} x, & \text{if } x \geq 0; \\ y, & \text{if } x < 0. \end{cases}$
 e. $f(x, y) = \begin{cases} \frac{xy+1}{xy}, & \text{if } xy \neq 0; \\ 0, & \text{if } xy = 0. \end{cases}$
 f. $f(x, y) = \sqrt{x^2 + y^2 + 1}$, $S = \mathbb{R}^2$.

EXERCISE 2.31 Consider $f(x, y) = x^2 + y^2 + 1$.

- Draw the intersection curve at $(x, y) = (0, 0)$ in the y -direction in a two-dimensional plot.
- Draw the intersection curve at $(x, y) = (0, 0)$ in the x -direction in a two-dimensional plot.
- Draw the level curve with a level of 10 in a two-dimensional plot.
- Show that $f(x, y) \geq 1$.
- Draw the intersection curve at $(x, y) = (0, 0)$ with the plane $y = x$ in a two-dimensional plot.
- Draw the intersection curve at $(x, y) = (0, 0)$ with the plane $y = \alpha x$ in a two-dimensional plot for $\alpha = 2$ and $\alpha = -1$.
- Based on the above results, make a plot of the graph of f in three-dimensional space.

EXERCISE 2.32 Determine which of the quadratic forms f , below, are either positive definite/positive semi-definite/negative definite/negative semi-definite.

- $f(x, y, z) = x^2 + 4xy + y^2$.
- $f(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.
- $f(x, y, z) = -x^2 - yz$.
- $f(x, y, z) = -x^2 + 2xy - 2y^2 + 4yz - 3z^2$.

Advanced, Deepening and Broadening Exercises

EXERCISE 2.33 Which of the next statements are true. Motivate your answer.

- Every function $f(x, y)$ attains a global maximum on $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$.
- Every convex function f has the property that f'' exists and is positive semi-definite on its domain.
- Matrix A is negative definite if and only if all leading principal minors are smaller than zero.
- Matrix A is positive definite if all principal minors are larger than zero.
- Matrix A is negative semi-definite if and only if matrix $-A + \epsilon I$ is positive definite for every $\epsilon > 0$.
- If both partial derivatives of $f(x, y)$ exist, $f(x, y)$ is differentiable.
- If $f(x, y)$ is differentiable, both partial derivatives exist.
- If $f(x, y)$ is differentiable, f is continuous.
- If $f(x, y)$ is continuous, both partial derivatives exist.
- If $f(x, y)$ is continuous, f is differentiable.
- Assume set S is not compact. Then every boundary point of S is an element of S .
- Assume S is a subset of $\{(x, y) \mid x^2 + y^2 \leq 100\}$ and every sequence s_n that converges to some limit s , where $s_i, i = 1, 2, \dots$ all belong to S , has the property that $s \in S$. Then $f(x, y) = x + y$ attains a global minimum on S .
- Assume $f(x, y)$ is twice continuous differentiable; has a stationary point at $(0, 0)$; and $Hf(0, 0) \geq 0$. Then f has a local minimum at $(0, 0)$.
- If both partial derivatives of $f(x, y)$ exist, $f(x, y)$ is continuous.

EXERCISE 2.34 (*Example of quadratic convex function that has not a minimum*)

Consider $f(x, y) = \frac{1}{2}(x + y)^2 + \alpha x + \beta y$, with $\alpha \neq \beta$.

- Show that f is convex.
- Show that f has no stationary points.
- Show that f has not a minimum.

EXERCISE 2.35 (*Cauchy-Schwarz and triangle inequality*)

Let $x, y \in \mathbb{R}^n$. Show that (you may restrict the proof to the case $n = 2$):

- $|x^T y| \leq \|x\| \|y\|$ (Cauchy-Schwarz inequality). Hint: Consider for a fixed x and y the quadratic polynomial $p(t) = \|x - ty\|^2$.
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality). Hint: Expand $\|x + y\|^2$ and use item a.
- Use item b. to show the so-called inverse triangle inequality: $|\|x\| - \|y\|| \leq \|x - y\|$. Hint: $x = x - y + y$.

EXERCISE 2.36 Assume f and g are continuous functions defined on open sets D_1 and D_2 , respectively.

Assume $f(x) = g(x)$ for all $x \in D_3 := D_1 \cap D_2$.

Consider the function $h(x) = \begin{cases} f(x) & \text{if } x \in D_1; \\ g(x) & \text{if } x \in D_2. \end{cases}$

Motivate why $h(x)$ is continuous on $D := D_1 \cup D_2$.

EXERCISE 2.37 (*Limits in $\mathbb{R}^n$, coordinate criterion*)

The intuitive idea that a sequence $\{s_n\}_{n=1}^\infty$, with $s_n \in \mathbb{R}^2$, approaches a vector $s \in \mathbb{R}^2$ (i.e., that the limit of $\{s_n\}_{n=1}^\infty$ is s) can be formally defined as follows: for every $\epsilon > 0$, no matter how small, there exists a number N_ϵ such that for all $n > N_\epsilon$, the distance between s_n and s , that is $\|s_n - s\|$, is less than ϵ .

Let $s_n := (x_n, y_n) \in \mathbb{R}^2$ and $s := (x, y)$. Use the above definition to show that

$$s_n \text{ approaches } s \text{ iff. } x_n \text{ approaches } x \text{ and } y_n \text{ approaches } y.$$

That is, a sequence in $\mathbb{R}^n$ has a limit iff. all its corresponding n coordinate sequences in $\mathbb{R}$ have a limit.

EXERCISE 2.38 (*Calculation rules with Limits*)

Let $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ with both $s_n, t_n \in \mathbb{R}^2$ be sequences that approach s and $t \in \mathbb{R}^2$, respectively. Define the sequence $\{u_n\}_{n=1}^\infty$ by $u_n := s_n + t_n$, $n = 1, 2, \dots$; and for a scalar λ , $\{v_n\}_{n=1}^\infty$ by $v_n := \lambda s_n$, $n = 1, 2, \dots$.

- Show that u_n approaches $s + t$ and v_n approaches λs .
- Give an example of a sequence $\{u_n\}_{n=1}^\infty$ with $u_n = s_n + t_n$, where u_n approaches zero, but both $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ don't have a limit.
- Assume $s_n, t_n, s, t \in \mathbb{R}$. Show that $w_n := s_n t_n$ approaches st .

- d. Assume $s_n, t_n, s, t \in \mathbb{R}$, where $t \neq 0$. Show that $x_n := \frac{1}{t_n}$ approaches $\frac{1}{t}$. Use this result and item c. to show that $y_n := \frac{s_n}{t_n}$ approaches $\frac{s}{t}$.

EXERCISE 2.39 (*combinations of continuous functions are continuous again*)

Let $f(x)$ and $g(x)$ be defined on an open set S , $x_0 \in S$, and λ and μ be any real numbers. Furthermore, let $k(x)$ be defined on $f(S)$ and continuous at $f(x_0)$. Show, using the definition, that if the functions $f(x)$ and $g(x)$ are continuous at x_0 , also next functions are continuous at x_0 .

- a. $\lambda f(x) + \mu g(x)$. b. $f(x)g(x)$. c. $\frac{f(x)}{g(x)}$, if $g(x_0) \neq 0$. d. $k \circ f(x)$.

EXERCISE 2.40 (*Proof Continuity property, Proposition 2.15*)

Let the continuous function f be defined on a connected set S . Let $I := f(S)$ and $s_i \in S$, $i = 1, 2$. Assume w.l.o.g. $f(s_1) \leq f(s_2)$ and $\lambda \in [f(s_1), f(s_2)]$. Use the Intermediate Value Theorem 1.18 to show that there exists a $s \in S$ such that $f(s) = \lambda$.

EXERCISE 2.41 Show, using the definition, that next functions are differentiable on $\mathbb{R}^2$.

- a. $f(x, y) = 1$. b. $f(x, y) = x$. c. $f(x, y) = xy$. (Hint: use $2\Delta x \Delta y \leq (\Delta x)^2 + (\Delta y)^2$.)

EXERCISE 2.42 (*Differentiability implies continuity*)

Using the definition of differentiability, show that if f is differentiable at (x_0, y_0) , this implies that $f(x_0 + \Delta x, y_0 + \Delta y)$ approaches $f(x_0, y_0)$ if Δx and Δy approach zero. Or, stated differently, f is continuous at x_0 .

EXERCISE 2.43 (*Mean Value Theorem*)

The line segment $[a, b]$ is defined as the set of points on the line through a and b , located between a and b : $\{x \mid x = (1 - \lambda)a + \lambda b, 0 \leq \lambda \leq 1\}$.

The Mean Value Theorem states the next property. Assume $f(x)$ is differentiable on an open convex set $S \subset \mathbb{R}^n$. Then, for every $a, b \in S$, there exists a point c on the line segment $[a, b]$, such that

$$f(b) - f(a) = Df(c)(b - a).$$

In item b., below, you are asked to prove this theorem.

- a. Consider the intersection function of $f(x)$ on the line segment $[a, b]$: $F(t) = f(a + t(b - a))$, $0 \leq t \leq 1$. Show that there exists a $\tau \in (0, 1)$ such that: $F(1) - F(0) = F'(\tau)(1 - 0)$.
- b. Use the chain rule of differentiation to conclude that there exists a point $c \in [a, b]$ such that $f(b) - f(a) = Df(c)(b - a)$.
- c. Consider $f(x, y) = e^{x-y}$. Show that for every two points (x_1, y_1) and (x_2, y_2) there exists a number $\lambda \in (0, 1)$, such that $e^{x_1-y_1} - e^{x_2-y_2} = e^{\lambda(x_1-y_1)+(1-\lambda)(x_2-y_2)}(x_1 - y_1 + y_2 - x_2)$.

EXERCISE 2.44 (combination of convex functions that are convex too)

Assume f, g are convex functions defined on some open convex set $S \subset \mathbb{R}^n$. Let $A \in \mathbb{R}^{n \times m}$ be an arbitrary matrix and $b \in \mathbb{R}^n$.

- Show that $\lambda f + \mu g$ is convex, if $\lambda \geq 0$ and $\mu \geq 0$.
- Show that $h(x) := f(Ax + b)$ is convex.
- Show that $h(x) = \max\{f(x), g(x)\}$ is convex.

EXERCISE 2.45 (Taylor's theorem for a twice differentiable function)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice differentiable on an open convex set $S \subset \mathbb{R}^n$. Taylor's theorem states, we can find for every x and a in S a point ζ that is located on the line segment between x and a such that,

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}(x - a)^T Hf(\zeta)(x - a). \quad (2.4)$$

In this exercise you are asked to provide a proof of this result.

- For a fixed a and x consider the line $y(t) = tx + (1 - t)a$, $t \in \mathbb{R}$. Show that $F(t) := f(y(t))$ is a twice differentiable scalar function on $\mathbb{R}$.
- Show that $F(1) = F(0) + F'(0) + \frac{1}{2}F''(t_0)$, where $t_0 \in (0, 1)$.
- Prove (2.4).

EXERCISE 2.46 Make a Taylor series expansion (2.4) for the functions f below, with $a = (0, 0)$ and $a = (1, 2)$, respectively.

- $f(x) = e^{x+y}$.
- $f(x) = \frac{y}{1+x}$.
- $f(x) = x^2 + xy + 2y^2 + 1$.

EXERCISE 2.47 (Taylor's theorem for a vector-valued twice differentiable function)

Let $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \dots, m$, be twice continuously differentiable on a neighborhood S of a . Assume, moreover, the second order derivatives of f_j are bounded on S , i.e. there exists a scalar d such that $-dI \leq Hf_j(x) \leq dI$, for all $x \in S$ and $j = 1, \dots, m$.

Let $f := [f_1 \cdots f_m]^T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Consider the notation $f(x) = O(x)$ as $x \rightarrow 0$, which is defined as that there exist scalars c and ϵ such that $\|f(x)\| \leq c\|x\|$ when $0 < \|x\| \leq \epsilon$.

Then Taylor's theorem states that, with $f'(a) := \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_m(a) \end{bmatrix}$:

$$f(x) = f(a) + f'(a)(x - a) + O(\|x - a\|^2). \quad (2.5)$$

In this exercise you are asked to provide a proof of this result.

- Show that for every f_j we can find a point ζ_j that is located on the line segment between x and a such that,

$$f_j(x) = f_j(a) + f'_j(a)(x - a) + \frac{1}{2}(x - a)^T Hf_j(\zeta_j)(x - a).$$

- b. Show that for every f_j , $|f_j(x) - (f_j(a) + f'_j(a)(x - a))| \leq \frac{1}{2}d\|x - a\|^2$, for all $x \in S$.
- c. Show that for all $x \in S$, $\|f(x) - (f(a) + f'(a)(x - a))\| \leq \frac{1}{2}\sqrt{md}\|x - a\|^2$. Conclude (2.5).

EXERCISE 2.48 (*Newton-Raphson's algorithm to find a stationary point of a multi-variable function*)

Assume the partial derivatives of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ are twice continuously differentiable on a neighborhood S of a and f has a stationary point at $x = a$. Assume, moreover, that $H(D_j f)$, $j = 1, \dots, n$, are bounded on S (i.e., there exists a d such that $-dI \leq H(D_j f(x)) \leq dI$ on S for all j). In this exercise we consider the multi-variable extension of Newton-Raphson's algorithm

$$x_{N+1} = x_N - Hf^{-1}(x_N)Df^T(x_N), \quad x_0 = \bar{x} \in S, \quad (2.6)$$

to produce successively better approximations of this stationary point a .

- a. Show that if $x_N \in S$ is an initial guess for the stationary point a and $x_{N+1} \in S$ an update of this guess, $Df^T(x_{N+1}) = Df^T(x_N) + Hf(x_N)(x_{N+1} - x_N) + O(\|x_{N+1} - x_N\|^2)$.
- b. Assume we are looking for an update of x_N that has the property that $Df^T(x_{N+1}) = 0$ and $Hf(x)$ is invertible on S . Show that the update (2.6) makes sense.
- c. Consider the Newton-Raphson approximations (2.6) of the stationary point a .

Let c be the absolute smallest eigenvalue of $Hf(x)$ on S (which implies that $Hf^{-1}(x)Hf^{-1}(x) \leq c^2I$ for all $x \in S$). Assume $Hf(x)$ satisfies a so-called Lipschitz condition $\|Hf(x) - Hf(y)\| \leq L\|x - y\|$ on S (with $c > 0$ and $L > 0$)⁸.

Show that with $\epsilon_N := \|x_N - a\|$, this update error satisfies: $\epsilon_{N+1} \leq \frac{1}{c}(L + \frac{1}{2}\sqrt{nd})\epsilon_N^2$, with $\epsilon_0 = \|\bar{x} - a\|$.

- d. Let $\tilde{\epsilon}_N := \frac{1}{c}(L + \frac{1}{2}\sqrt{nd})\epsilon_N$. Show that $\tilde{\epsilon}_{N+1} \leq \tilde{\epsilon}_N^2$ with $\tilde{\epsilon}_0 = \frac{1}{c}(L + \frac{1}{2}\sqrt{nd})\|\bar{x} - a\|$.

- e. Use item d to conclude that $\tilde{\epsilon}_{N+1} \leq \tilde{\epsilon}_0^{2^N}$.

Assume $\frac{1}{c}(L + \frac{1}{2}\sqrt{nd})\epsilon_0 < 1$. Show that x_n in the Newton-Raphson algorithm (2.6) converges to the stationary point a .

- f. Formulate a recursive Newton-Raphson procedure that will provide the location (x, y) at which the function $f(x, y) = x^4 - 4x + y^4 - 32y$ attains its minimum.

EXERCISE 2.49 (*Stationary point of a convex function is a global minimum*)

In this exercise we use Taylor's theorem (2.4) to show that if a function f is convex, twice continuous differentiable on an open convex set S and has a stationary point at $x = a$, $f(a)$ is a global minimum.

- a. Show that if f has a stationary point at $x = a$ there exists a point ζ , that is located somewhere between x and a , such that $f(x) = f(a) + \frac{1}{2}(x - a)^T Hf(\zeta)(x - a)$.
- b. Use item a to show that if f is convex, $f(x) \geq f(a)$.
- c. Show that $f(x, y) = (x - 1)^4 + y^2 - 2y - 4$ has a global minimum at $(1, 1)$.

⁸ Here we use the 2-norm as matrix norm. That is $\|A\|$ is the largest singular value of matrix A , i.e., $\|A\|$ is the square root of the largest eigenvalue of matrix $A^T A$. Using this matrix norm one also easily verifies the inequality $\|Ax\| \leq \|A\|\|x\|$ applies.

d. Show that $f(x, y) = (x - y)^2$ has a global minimum at all points (z, z) , $z \in \mathbb{R}$.

EXERCISE 2.50 Assume f and g are continuous functions on $\mathbb{R}^2$. Consider the function $m(x, y) = \text{minimum}\{f(x, y), g(x, y)\}$ for all $(x, y) \in \mathbb{R}^2$. Motivate why $m(x, y)$ attains both a global maximum and a global minimum on $S = \{(x, y) \mid |x| \leq 1 \text{ and } |y| \leq 1\}$.

EXERCISE 2.51 (Proof Theorem 2.28)

Prove Theorem 2.28, item i) and ii), using the definition of differentiability. Hint: for item ii), add and subtract into the definition the term $f(x_0 + \Delta x, y_0 + \Delta y)g(x_0, y_0)$ and next recollect terms.

EXERCISE 2.52 (Example of function which partial derivatives exist, but is not differentiable)

Consider $f(x, y) = \begin{cases} x, & \text{if } y = 0; \\ 0, & \text{if } y \neq 0. \end{cases}$

- Consider the section function $f(x, 0)$. Show that $\frac{\partial f}{\partial x}(0, 0) = 1$.
- Consider the section function $f(0, y)$. Show that $\frac{\partial f}{\partial x}(0, 0) = 0$.
- Motivate why $f(x, y)$ is not differentiable at $(0, 0)$.
- Let $\epsilon \neq 0$. Motivate why $\frac{\partial f}{\partial y}(\epsilon, 0)$ does not exist.
- Motive why Theorem 2.26 does not apply here.

EXERCISE 2.53 (Example of function which intersection function on any line segment through its stationary point has a minimum, but has not a minimum)

Consider $f(x, y) = (y - x^2)(y - 3x^2)$.

- Show that f has a stationary point at $(0, 0)$.
- Show that $f''(0, 0)$ is positive semi-definite.
- Consider the section function $f(x, kx)$, where k is an arbitrary real number. Show that for every k , this section function has a minimum at $x = 0$.
- Consider the section function $f(0, y)$. Show that this section function has a minimum at $y = 0$.
- Show that $f(x, y)$ has a saddle point at $(0, 0)$.

EXERCISE 2.54 (Example of function which is continuous but which partial derivatives don't exist) Con-

sider $f(x, y) = \begin{cases} |x|, & \text{if } y = 0; \\ |y|, & \text{if } x = 0; \\ 0, & \text{elsewhere.} \end{cases}$

- Consider the section function $f(x, 0)$. Show that $\frac{\partial f}{\partial x}(0, 0)$ does not exist.
- Consider the section function $f(0, y)$. Show that $\frac{\partial f}{\partial x}(0, 0)$ does not exist.
- Motivate why $f(x, y)$ is continuous at $(0, 0)$.

EXERCISE 2.55 (Example of function that is differentiable, but which partial derivatives are not continuous)

Consider $f(x, y) = (x^2 y^2)^{\frac{1}{3}}$.

- Show that both partial derivatives of $f(x, y)$ exist at $(0, 0)$.
- Calculate $D_1 f(x_0, y_0)$ at all (x_0, y_0) for which $x_0 y_0 \neq 0$.
- Show that $D_1 f(x, y)$ is not continuous at $(0, 0)$.
- Show that $f(x, y)$ is differentiable at $(0, 0)$.

EXERCISE 2.56 (Extension of notion of differentiability)

Let $f(x)$ be defined on a subset C of $\mathbb{R}^n$ and $x_0 \in C$. Then f is said to be *differentiable at x_0* if there exists a vector $a \in \mathbb{R}^n$ (where a may depend on x_0) such that for all vectors $h \neq 0$ such that $x_0 + h \in C$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) - a^T h}{\|h\|} = 0.$$

If this limit exists, a is called the derivative of f at x_0 .

Notice there are no restrictions here on the set C and x_0 .

- Consider $f(x, y) = |y|$ on the constraint set $C = \{(x, y) \mid y = x^2\}$. Show that f is differentiable at $(0, 0)$ on C .
- Consider $f(x, y) = |y|$ on the constraint set $C = \{(x, y) \mid x = y^2\}$. Show that f is not differentiable at $(0, 0)$ on C .
- Consider $f(x, y) = |y|$ on the constraint set $C = \{(x, y) \mid y = |x|\}$. Show that f is not differentiable at $(0, 0)$ on C .
- Consider $f(x, y) = y$ on the constraint set $C = \{(x, y) \mid x^2 = y^2\}$. Show that f is not differentiable at $(0, 0)$ on C .

EXERCISE 2.57 (Brouwer's Fixed Point Theorem)

Let $S \subset \mathbb{R}^2$ be a compact, convex set (nonempty). Assume $f_i : S \rightarrow \mathbb{R}$, $i = 1, 2$, are continuous functions. Consider the range of f defined as: $\{(f_1(x, y), f_2(x, y)) \in \mathbb{R}^2 \mid (x, y) \in S\}$. Then Brouwer's Fixed Point Theorem states that if the range of f is a subset of S , there exists $(\bar{x}, \bar{y}) \in S$ such that $f_1(\bar{x}, \bar{y}) = \bar{x}$ and $f_2(\bar{x}, \bar{y}) = \bar{y}$. $(\bar{x}, \bar{y})$ is called a fixed point of f in that case. (This result generalizes for n functions which domain S is a compact, convex set in $\mathbb{R}^n$).

Unfortunately, currently, a proof using elementary arguments we developed so far of this result is lacking in literature. Below, you are asked to show one of the many applications Brouwer's theorem has. Here we consider the issue to find a stationary point.

- Let S be a compact, convex set in $\mathbb{R}^2$. Show that a continuous differentiable function $f : S \rightarrow \mathbb{R}$ has a stationary point $(\bar{x}, \bar{y})$ if and only if the function $g(x, y) := (D_1 f(x, y) + x, D_2 f(x, y) + y)$ has a fixed point on S .
- Consider the function $f(x, y) = \sqrt{1 + \frac{1}{2}x^2 y^2} + e^{-\frac{1}{4}(x^2 + y^2)} - \frac{1}{2}(x^2 + y^2)$. Show that f has a stationary point within the set $S = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$.

EXERCISE 2.58 (*Fixed Point Contraction Mapping Theorem*)

Let $S \subset \mathbb{R}^n$ be a (nonempty) closed convex set and $f_i : S \rightarrow \mathbb{R}$, $i = 1, \dots, n$. Consider the function $f := (f_1, \dots, f_n)$ on S . Assume $f(x) \in S$ for all $x \in S$ and, moreover, there exists a constant $0 \leq c < 1$ such that for all $x, y \in S$, $\|f(x) - f(y)\| \leq c\|x - y\|$. Then the Contraction Mapping Theorem states that under these conditions f has a *unique* fixed point $\bar{x} \in S$.

Furthermore, $\bar{x}$ can be found as follows: start with an arbitrary element $x_0 \in S$ and define a sequence $\{s_n\}_{n=1}^\infty$ by $s_n = f(s_{n-1})$, $s_1 = x_0$, then s_n converges to $\bar{x}$.

In items a. and b., below, you are asked to provide a proof of this result under the additional assumption that S is bounded and f is continuous.

- Assume f is continuous and S compact. Show that f has a fixed point on S .
- Show that f has a unique fixed point on S . Hint: assume there is another one.
- Let $x \in S$. Consider the sequence $\{s_n\}_{n=1}^\infty$ defined by: $s_1 = x$ and $s_{n+1} = f(s_n)$, $n = 2, 3, \dots$. Show that $\|f(s_n) - s_n\| \leq c^n \|s_2 - s_1\|$. (Using this, one can show that this implies that f must have a fixed point, i.e., that the Contraction Mapping Theorem applies also without assuming continuity of f and compactness of S).
- Consider the function $f(x, y) = (\frac{1}{2}y + 3, \frac{1}{2}x)$. Show, using the Contraction Mapping Theorem, that f has a unique fixed point on $\mathbb{R}^2$. Consider $s_1 := (0, 0)$. Calculate $s_{n+1} = f(s_n)$, for $n = 2, 3, 4, 5$. Calculate the fixed point.
- Show, using the Contraction Mapping Theorem, that $f(x, y) = \frac{1}{2} \ln(2 + x^2 + y^2) - \frac{1}{2}(x^2 + y^2)$ has a unique stationary point. Hint: consider the approach used in Exercise 2.57, item a. Use the Mean Value Theorem to estimate the involved partial derivatives.

Exercises with an Economic Content**EXERCISE 2.59** (*generalized Cobb-Douglas production function*)

Show that the production function $F(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$, with $0 < \alpha_i < 1$ is concave on its domain $D = \{(x_1, x_2, x_3) \mid x_1 > 0, x_2 > 0, x_3 > 0\}$.

EXERCISE 2.60 A country i considers entering an economic union with N members (including the country itself). Welfare of country i equals $U_i = y_i - g_i + \alpha_i h(g_i + \beta \sum_{j \neq i} g_j)$, where $y_i - g_i$ is the amount of private consumption and g_i is the local public good level in country i . Here, $h(x)$ is an increasing function. The *first-best union* is the union where the sum of welfare of the individual countries is maximized.

- Determine the first-order conditions for the public good for the first-best union if $N = 3$.
- Let $h(x) = \ln(x)$. Solve the first-order conditions you found in item a. Hint: introduce $x_i = \frac{\alpha_i}{g_i + \beta \sum_{j \neq i} g_j}$ and rewrite the system of equations in terms of x_i .

- c. Solve the first-order conditions in item b for an arbitrary number of countries.

EXERCISE 2.61 (*optimal carbondioxide emissions*)

Consider N countries who want to optimize CO_2 emissions. Welfare of country i equals: $W_i = R_i(e_i) - D_i(\sum_{j=1}^N e_j)$, where R_i are welfare revenues associated with the production of emissions e_i and D_i is the associated pollution damage. All functions are assumed to be twice continuous differentiable, $R_i(x)$ concave, and $D_i(x)$ convex. Assume countries agree to maximize total welfare in the countries.

- Consider $N = 3$. Show that, at the optimum (e_1^*, e_2^*, e_3^*) , necessarily $R'_i(e_i^*) = R'_j(e_j^*)$, $i, j = 1, 2, 3$.
- Show that the total welfare function is a concave function.
- Assume the welfare function has a stationary point (e_1^*, e_2^*, e_3^*) . Show that there exists a unique set of emissions, given by (e_1^*, e_2^*, e_3^*) , where total welfare is maximized.
- Consider $N = 3$, with $R_i(x) = 2\alpha_i\sqrt{x}$ and $D_i(x) = \beta_i x$, where $\alpha_i > 0$ and $\beta_i > 0$. Determine the optimal emissions. Show that this optimum indeed yields a maximum.

EXERCISE 2.62 A producer's production function of shavers is given by $Y(K, L) = 2K^{\frac{1}{2}}L^{\frac{1}{4}}$. The cost of one unit of capital, K , is 8 and of one unit of labour, L , is 2. The market price of shavers is 32.

- Show that the profit function of the producer is $\pi(K, L) = 64K^{\frac{1}{2}}L^{\frac{1}{4}} - 8K - 2L$.
- Show that $\pi(K, L)$ is a concave function on the domain $K > 0$ and $L > 0$.
- Show that $\pi(K, L)$ has a stationary point at $(K, L) = (512, 1024)$.
- Determine the maximum profits the producer can make.

EXERCISE 2.63 A baker can produce both bread, b , and cookies, c . His cost function is $C(b, c) = \frac{2}{3}b^2 + \frac{1}{3}bc + \frac{1}{6}c^2$. The market price of bread is 3 and of cookies is 1.

- Determine the profit function of the producer.
- Determine the amount of bread and cookies that maximize his profit.
- Determine the maximum profits the producer can make.

EXERCISE 2.64 A travel agency has a fixed budget of 14 (in millions of euro) to advertize a cruise. It has three gremia for this purpose: newspaper, television and internet. Let x_1 denote the amount of money spent on advertizing in the newspaper, x_2 the amount of money spent on advertizing on the television and x_3 the amount of money spent on advertizing on the internet.

The number of customers N_j , $j = n, t, i$, when (x_1, x_2, x_3) is spent on advertizing is:

$$N_n(x_1, x_2, x_3) = 2x_1 + x_2 + x_3; \quad N_t(x_1, x_2, x_3) = x_1 + 3x_2 + x_3; \quad N_i(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3.$$

Assume the price of the cruise is p .

- Show that the number of customers is determined by maximizing

$$f(x_1, x_2) = 2x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + \frac{3}{2}(x_1 + x_2)x_3, \text{ where } x_3 = 100 - x_1 - x_2.$$

b. Determine how much of the price p consists of advertising cost.

EXERCISE 2.65 A farmer has 100 (acres) of land. He has to decide how much land he wants to use to grow corn, potatoes and wheat, respectively. On x (acres) of land he can produce $3x$, $6x$ and $2x$ kilograms (in hundreds) of corn, potatoes and wheat, respectively. The cost involved of growing x (acres) of corn, potatoes and wheat are $\ln(1+3x)$, $2\ln(1+6x)$ and $2\ln(1+2x)$, respectively. The price of selling 100 kilograms of corn, potatoes and wheat is 2, 1 and 2 euros (in hundreds), respectively.

- Show that the profit function of the farmer is given by $\Pi(x_1, x_2, x_3) = 6x_1 + 6x_2 + 4x_3 - \ln(1 + 3x_1) - 2\ln(1 + 6x_2) - 2\ln(1 + 2x_3)$ where x_i , $i = 1, 2, 3$, is the land used for growing corn, potatoes and wheat, respectively, and $x_3 = 100 - x_1 - x_2$.
- Determine how much acres the farmer has to use for growing corn, potatoes and wheat, respectively, in order to maximize his profits.

EXERCISE 2.66 An investor wants to decide how much of his capital he wants to put on the bank, earning a risk-free interest rate, $r = 0.03$; investing it in stock 1, getting an expected return of $\mu_1 = .04$, at a risk $\sigma_1 = .0316$; or investing it in stock 2, getting an expected return of $\mu_2 = .08$, at a risk $\sigma_2 = .05$. Let the fractions he invests in stock 1 and 2 and he puts on the bank be ω_i , $i = 1, 2, 3$, respectively. Assume the expected return of such a portfolio is $\mu = .04\omega_1 + .08\omega_2 + .03\omega_3$ and the risk $\sqrt{0.001\omega_1^2 + 0.0025\omega_2^2 - 0.002\omega_1\omega_2}$. Assume the investor likes to maximize his utility function $U := \mu - 100\sigma^2$.

- Show that $U(\omega_1, \omega_2, \omega_3) = -0.1\omega_1^2 - 0.25\omega_2^2 + 0.2\omega_1\omega_2 + 0.04\omega_1 + 0.08\omega_2 + 0.03\omega_3$. where $\omega_3 = 1 - \omega_1 - \omega_2$.
- Determine the fractions he should invest in the stocks in order to maximize his utility $U(\omega_1, \omega_2)$. What is the fraction he should invest on the bank?

EXERCISE 2.67 (*optimal pollution*)

Let $U(y, p)$ measure the total welfare of society, where y is the number of produced goods and p the associated level of pollution. Assume that welfare increases if y increases, and decreases if p increases.

- Show there exists not an optimal level of welfare under the above model assumptions.
- Substantiate why the above model might not be a correct model of reality.
- Consider the variable $z = \frac{p}{y}$. Motivate why for large values of z utility, $U(z)$, might be negative.
- Consider $U(z) = -\frac{1}{2}z^2 + z$. Determine the optimal ratio between pollution and goods production that maximizes welfare.

EXERCISE 2.68 (*homogeneous functions*)

The production function $Y(K, L)$ is called homogeneous of degree k if a multiplication of both inputs K and L by a factor t leads to a multiplication of the output Y of t^k . That is, if $Y(tK, tL) = t^k Y(K, L)$.

- Consider the Cobb-Douglas production function $Y(K, L) = cK^\alpha L^\beta$, where c, α and β are positive constants. Show that $Y(K, L)$ is homogeneous of degree 1 if and only if $\alpha + \beta = 1$.

- b. Explain in item a. what the effect on production is of doubling the amount of K and L if the degree of homogeneity is 1, smaller than 1 and larger than 1, respectively.
- c. Consider $f(t) = Y(tK, tL)$. Show that $f'(t) = D_1Y(tK, tL)K + D_2Y(tK, tL)L$.
- d. Consider $f(t) = t^k Y(K, L)$. Show that $f'(t) = kt^{k-1}Y(K, L)$.
- e. Assume $Y(K, L)$ is homogeneous of degree k . Show that $DY(K, L) \begin{bmatrix} K \\ L \end{bmatrix} = kY(K, L)$. Hint: use item c. and d. with $t = 1$.
- f. Use item e. to show that if a homogeneous function has degree k : $D_1Y(K, L)\frac{K}{Y(K, L)} + D_2Y(K, L)\frac{L}{Y(K, L)} = k$. (In economic terminology: the sum of the partial elasticities equals the degree of homogeneity).
- g. Consider the Cobb-Douglas production function from item a., with $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$. Verify the correctness of item f.

EXERCISE 2.69 Consider a producer of a product who can sell this product on two separate markets. The price he gets on both markets depends on the amount he supplies to that market. For market 1 this relationship is given by $p(x) = 20 - \frac{1}{2}x$, whereas for market 2 it is given by $p(x) = 25 - 2x$. Assume the cost of producing x units of the product is $C(x) = 5x + 60$.

- a. Determine the profit function $\Pi(x_1, x_2)$ for this producer, where x_i denotes the amount of products sold on market i , $i = 1, 2$.
- b. Calculate his maximal profit and the amounts of the product he will sell on both markets.
- c. Assume the producer is not allowed to ask different prices on both markets. Calculate his profit under this additional constraint. What are his losses compared to the case that price discrimination is allowed.

EXERCISE 2.70 Consider the Cob-Douglas production function $Y(K, L) = K^{\frac{2}{3}}L^{\frac{2}{3}}$, where K and L denote the amount of capital and labour used, respectively. Assume the price of capital is $\frac{1}{3}$ and of labour is $\frac{1}{3}$ too. Assume the price the producer gets for his product depends as follows on his production $p = 3Y^{-\frac{1}{2}}$.

- a. Determine the profit function, $\Pi(K, L)$ for this producer.
- b. Calculate his maximal profit and the corresponding production.

EXERCISE 2.71 (*profit maximization by a firm*)

Assume the production function of a firm is $Y(K, L)$, where K and L denote the amount of capital and labour used, respectively. Assume the price of capital is r and of labour is w . Assume the price the firm gets for his product is p .

- a. Assume p is fixed. Show that maximum profits are obtained only where next two equalities hold $Y'_K = \frac{r}{p}$ and $Y'_L = \frac{w}{p}$.

- b. Assume p depends on the production Y , i.e. $p = P(Y)$. Show that maximum profits are obtained only where next two equalities hold $Y'_K(P'Y + P) = r$ and $Y'_L(P'Y + P) = w$. Show that if profits are maximized then: $\frac{r}{Y'_K} = \frac{w}{Y'_L}$. Verify that these conditions are satisfied in Exercise 2.70.
- c. Assume p depends directly on the inputs K and L used for production, i.e. $p = P(K, L)$. Show that maximum profits are obtained only where next two equalities hold $\frac{P'_K}{P} + \frac{Y'_K}{Y} = \frac{r}{PY}$ and $\frac{P'_L}{P} + \frac{Y'_L}{Y} = \frac{w}{PY}$. Verify that these conditions are satisfied in Exercise 2.70, using $P(K, L) = 3K^{-\frac{1}{3}}L^{-\frac{1}{3}}$.

EXERCISE 2.72 (Price Discrimination)

Consider a producer of a product who can sell this product on two separate markets. The price he gets on both markets depends on the amount, x_i , $i = 1, 2$, he supplies to that market. Assume the cost of producing x units of the product is $C(x)$.

- a. Show that the profit function $\Pi(x_1, x_2)$ for this producer is $\Pi(x_1, x_2) = P_1(x_1)x_1 + P_2(x_2)x_2 - C(x_1 + x_2)$, where $P_i(x_i)$ is the inverse demand function for the product in market i , $i = 1, 2$.
- b. Show that maximum profits are obtained only where next two equalities hold $P'_i(x_i)x_i + P_i(x_i) = C'(x_1 + x_2)$, $i = 1, 2$. Interpret this equality in terms of profits obtained in both markets.
- c. Verify the conditions obtained in item b for Exercise 2.69.

CHAPTER 3

Constrained Optimization: Equality Constraints

In many (economic) optimization problems, the variables cannot be chosen freely. For example, when a consumer seeks to maximize utility (s)he is typically constrained by a budget. In this section, we assume that the optimizer faces a set of equality constraints on the variables while attempting to maximize the objective function. That is, the optimization problem can be stated as follows.

Problem Statement Let $f, g_i, i = 1, \dots, m$, be functions defined on an open subset $C \subset \mathbb{R}^n$ and $g := [g_1 \dots g_m]^T$.

$$\text{Optimize } f(s), \text{ subject to } g(s) = b, \text{ where } s \in \mathbb{R}^n \text{ and } g : C \rightarrow \mathbb{R}^m. \quad (3.1)$$

Note that, since we assume that the function $g(s)$ has m coordinate functions, above problem statement models the optimization of a single criterion subject to m equality constraints. If $S = \{x \in \mathbb{R}^n \mid g_i(x) = b_i, i = 1, \dots, m\}$ the set $D := C \cap S$ is called the *constraint set* or *feasible region*. We will assume throughout that this set is not empty. Within the feasible set defined by the constraints, we look for points at which f achieves a (local) maximum or minimum. These points are formally defined as follows.

DEFINITION 3.1 Let f be defined on an open set $C \subset \mathbb{R}^n$ and D be the feasible region of the constrained optimization problem. let $s_0 \in D$. Then, we say that f has a *local maximum (minimum)* at s_0 if $f(s) \leq (\geq) f(s_0)$ for all values of s in the intersection of some ball B with center s_0 and D . Furthermore, f has a *strict local maximum (minimum)* at s_0 if $f(s) < (>) f(s_0)$ for all $s \neq s_0 \in D \cap B$. We say that f has a *global maximum (minimum)* at s_0 if $f(s) \leq (\geq) f(s_0)$ for all $s \in D$.

Optimizing a function $f(s)$ subject to the constraint $g(s) = b$ is essentially nothing else than optimizing $f(s)$ over a restricted domain consisting of all points s that satisfy $g(s) = b$. In other words, we are

simply looking for the optima of a function whose domain is implicitly defined. Figure 3.1 gives a visual illustration of this basic idea.

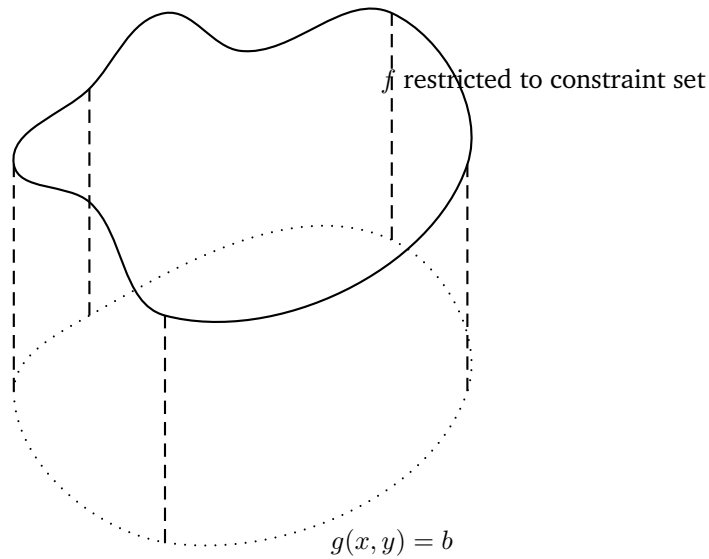


FIGURE 3.1 – *Constrained optimization.*

EXAMPLE 3.2 Consider the optimization of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g_1(x, y, z) = x + z = 1$ and $g_2(x, y, z) = y - z = 1$. Using the above notation,

$$g(x, y, z) = \begin{bmatrix} g_1(x, y, z) \\ g_2(x, y, z) \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note, from the equality constraints, that we can solve explicitly both x and y in terms of the variable z , i.e., $x = 1 - z$ and $y = 1 + z$, respectively. Substitution of this into the objective function, $f(x, y, z)$, shows that our optimization problem is equivalent with solving the scalar optimization problem

$$F(z) = (1 - z)^2 + (1 + z)^2 + z^2, \text{ with } x = 1 - z \text{ and } y = 1 + z.$$

Differentiation of F yields $F'(z) = 4 + 2z$. So, F has a stationary point at $z = -2$. Since $F''(-2) = 2 > 0$, we conclude that F has a minimum at $z = -2$. So, our optimization problem has a minimum at $(x, y, z) = (3, -1, -2)$.

The previous example essentially illustrates the basic procedure for solving Problem (3.1) analytically. That is, from the system of equations $g(s) = b$, we solve for m variables in terms of the remaining $n - m$ variables, either explicitly or implicitly. We then substitute these m variables into the objective function f , reducing it to a function of only the $n - m$ remaining variables. Finally, we optimize this reduced objective function - now unconstrained - using the theory developed in Chapter 2. Geomet-

rically, as illustrated in Figure 3.1, this process corresponds to explicitly describing the intersection function and then finding its optima.

If it is possible to explicitly solve the system of equations $g(s) = b$ for m variables in terms of the remaining $n - m$ variables, the procedure outlined above can be carried out in practice. When done this way, the problem is said to be solved using the *the substitution method*. However, if the constraints involve more complicated functions, this method may fail. Fortunately, under certain mild assumptions on the functions $g(s)$, it can be shown that even if these relationships cannot be determined explicitly, they exist at least theoretically. This theoretical knowledge is sufficient to proceed in a similar way. The resulting optimality conditions can then be elegantly expressed by introducing new variables, known as Lagrange multipliers. Solving the problem using these auxiliary variables is referred to as *the method of Lagrange*.

Both methods have their pros and cons. The substitution method has the benefit that results are often obtained more easily, but its applicability is limited. The method of Lagrange, on the other hand, provides a convenient and systematic way to solve constrained optimization problems, and the resulting Lagrange multipliers admit a clear economic interpretation. Its main drawback is that it yields only first-order conditions; consequently, whether a point satisfying these conditions corresponds to a local (or, even global) optimum often remains an open question. We will also discuss second-order conditions, which, however, are more involved to formulate.

3.1 The Substitution Method

We will illustrate the substitution method through a number of examples.

EXAMPLE 3.3 Consider the optimization of $f(x, y) = 9 + x^2 - x^2y^2 - 4\frac{1}{2}y^2$ subject to the constraint $g(x, y) = y - 6x = 0$.

From the equality constraint we obtain $y = 6x$. Substituting this into the objective function $f(x, y)$, we see that the problem is equivalent to the scalar optimization problem

$$F(x) = 9 + x^2 - 40\frac{1}{2}x^4, \text{ with } y = 6x.$$

Differentiating F gives

$$F'(x) = 2x - 162x^3 = 2x(1 - 81x^2).$$

Hence, F has stationary points at $x = 0$, $x = \frac{1}{9}$, and $x = -\frac{1}{9}$.

Next, consider the second-order derivative,

$$F''(x) = 2 - 486x^2.$$

Since $F''(-\frac{1}{9}) = F''(\frac{1}{9}) < 0$, F has a maximum at both of these points. Moreover, since $F''(0) > 0$, F has a minimum at $x = 0$.

So, the function f attains maxima at $(x, y) = (\frac{1}{9}, \frac{2}{3})$ and $(x, y) = (-\frac{1}{9}, -\frac{2}{3})$, and a minimum at $(x, y) = (0, 0)$.

To determine whether these extrema are local or global, we examine the sign scheme of F' , shown in Figure 3.2. From this diagram, it is clear that both maxima are global (since $F(\frac{1}{9}) = F(-\frac{1}{9})$), while the minimum at $x = 0$ is only local (since, for example, $F(10) < F(0)$).

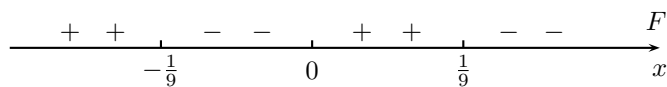


FIGURE 3.2 – Sign scheme of the derivative of F in Example 3.3.

Next Example 3.4 illustrates that the reduction of the constrained optimization problem to a lower order optimization problem may bring on restrictions on the considered domain for the implied optimization problem. Thus, when using the substitution method, one must always be mindful of and carefully consider the domain of the reduced optimization problem.

EXAMPLE 3.4 Consider the optimization of $f(x, y) = x^2 + y^2 - yx^2$ subject to the constraints $g(x, y) = x^2 + y^2 = 3$.

From the equality constraint, we obtain $x^2 = 3 - y^2$. Note, however, that the constraint also imposes a constraint on the domain of the variable y : specifically, y must lie in the interval $[-\sqrt{3}, \sqrt{3}]$. Substituting $x^2 = 3 - y^2$ into the objective function shows that the original constrained optimization problem is equivalent to the following scalar optimization problem:

$$F(y) = 3 - y(3 - y^2), \quad -\sqrt{3} \leq y \leq \sqrt{3}, \quad \text{with } x^2 = 3 - y^2.$$

Differentiating F with respect to y yields

$$F'(y) = -3 + 3y^2.$$

Hence, F has stationary points at $y = 1$ and $y = -1$. To determine what kind of extremum is attained at the different stationary points, we examine the sign scheme of the derivative of F , as illustrated in Figure 3.3.

From the sign diagram of F' , we conclude that F attains a boundary minimum at $y = -\sqrt{3}$, a local maximum at $y = -1$, a local minimum at $y = 1$ and a boundary maximum at $y = \sqrt{3}$. Note that $F(-\sqrt{3}) = F(\sqrt{3}) = 3$, $F(-1) = 5$ and $F(1) = 1$. Consequently, f has a global maximum at the points $(x, y) = (\sqrt{2}, -1)$ and $(x, y) = (-\sqrt{2}, -1)$, and a global minimum at $(x, y) = (\sqrt{2}, 1)$ and

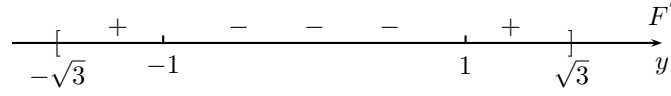


FIGURE 3.3 – Sign scheme of the derivative of F in Example 3.4.

$(x, y) = (-\sqrt{2}, 1)$. In addition, there is a local maximum at $(x, y) = (0, \sqrt{3})$ and a local minimum at $(x, y) = (0, -\sqrt{3})$.

EXAMPLE 3.5 Consider the optimization of $f(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$ subject to the constraints $g_1(x_1, x_2, x_3, x_4) = x_1 + x_3 = 1$ and $g_2(x_1, x_2, x_3, x_4) = x_2 + x_4 = 1$.

From the first equality constraint, g_1 , we have that $x_3 = 1 - x_1$. The second equality constraint, g_2 , yields $x_4 = 1 - x_2$. Substitution of this into the objective function, f , shows that our optimization problem is equivalent with solving the multi-variable optimization problem

$$F(x_1, x_2) = x_1 x_2 (1 - x_1)(1 - x_2), \text{ where } x_3 = 1 - x_1 \text{ and } x_4 = 1 - x_2.$$

Note that the domain of F is $\mathbb{R}^2$. Differentiation of F yields

$$\begin{aligned} F' &= [x_2(1 - x_1)(1 - x_2) - x_1 x_2(1 - x_2) \quad x_1(1 - x_1)(1 - x_2) - x_1 x_2(1 - x_1)] \\ &= [x_2(1 - 2x_1)(1 - x_2) \quad x_1(1 - x_1)(1 - 2x_2)]. \end{aligned}$$

So, the stationary points of F are $(\frac{1}{2}, \frac{1}{2})$; $(0, 0)$; $(1, 0)$; $(0, 1)$ and $(1, 1)$. To see whether these stationary points are optima, we calculate F'' :

$$F'' = \begin{bmatrix} -2x_2(1 - x_2) & (1 - 2x_1)(1 - 2x_2) \\ (1 - 2x_1)(1 - 2x_2) & -2x_1(1 - x_1) \end{bmatrix}.$$

So,

$$\begin{aligned} HF(\frac{1}{2}, \frac{1}{2}) &= \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}; \quad HF(0, 0) = HF(1, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \\ HF(1, 0) &= HF(0, 1) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \end{aligned}$$

Since the Hessian matrix $HF(\frac{1}{2}, \frac{1}{2})$ is negative definite, F has a local maximum at $(\frac{1}{2}, \frac{1}{2})$. It can be readily verified that all other stationary points are saddle points. Note that $F(-1, -1) = 4 > F(\frac{1}{2}, \frac{1}{2}) = \frac{1}{16}$. Therefore, F does not attain a global maximum at $(\frac{1}{2}, \frac{1}{2})$.

Consequently, the function f has a local maximum at $(x_1, x_2, x_3, x_4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

EXERCISE 3.1 Determine all optima (location, value, kind (local/global)) of the next constrained optimization problems.

- Optimize $f(x, y) = xy^2$, subject to the constraint $g(x, y) := x + y = 1$.
- Optimize $f(x, y) = e^x y^2$, subject to the constraint $g(x, y) := x + y^2 = 1$.
- Optimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraints $g_1(x, y, z) := x + y + z = 0$ and $g_2(x, y, z) := x - y + z = 2$.
- Optimize $f(x, y, z) = \frac{1}{2}x^2 y + z$ subject to the constraints $g_1(x, y, z) := \frac{1}{2}x^2 + 1 - y = 0$ and $g_2(x, y, z) := \frac{4}{3}x - \frac{4}{3}xy - z = 0$.
- Optimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraint $g(x, y, z) := x - y + 2z = 6$.
- Optimize $f(x, y, z) = e^x + e^{3-y} + e^z$, subject to the constraint $g(x, y) := x - y + z = 0$.

EXERCISE 3.2 Determine all optima (location, value, kind (local/global)) of the next constrained optimization problems.

- Optimize $f(x, y) = xy^2$, subject to the constraint $g(x, y) := x^2 + 2y^2 = 3$.
- Optimize $f(x, y) = \frac{x}{y} + \frac{y}{x}$, subject to the constraint $g(x, y) := xy = 1$.
- Optimize $f(x, y) = \frac{1}{3}y^2 e^x - y^2$, subject to the constraint $g(x, y) := e^x - y = -1$.
- Optimize $f(x, y, z) = xz^2 - 7x - 10y + 6xy$ subject to the constraints $g_1(x, y, z) := x^2 + y^2 + z^2 = 9$ and $g_2(x, y, z) := x - y = 0$.
- Optimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraints $g_1(x, y, z) := x^2 + y^2 = 4$ and $g_2(x, y, z) := x^2 + z^2 = 9$.
- Optimize $f(x, y, z) = (z - 3)^2 - xy$, subject to the constraint $g_1(x, y, z) := (x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9$ and $g_2(x, y, z) := x + y = 0$.

3.2 Two variables and one constraint: a Geometric Approach

Let f and g be two functions defined on an open subset $C \subset \mathbb{R}^2$. In this section we consider the constrained optimization problem

$$\text{optimize } f(x, y) \text{ subject to } g(x, y) = b. \quad (3.2)$$

We provide a geometric approach to solve the problem. This approach ultimately leads to the introduction of the so-called Lagrange function to solve the optimization problem.

As an example consider the problem to optimize

$$f(x, y) = xy \text{ subject to } g(x, y) = x^2 + y^2 = 4,$$

where, for simplicity, we assume that both $x > 0$ and $y > 0$.

Geometrically, the constraint set corresponds to the part of a circle with center $(0, 0)$ and radius 2 that

lies in the positive orthant. Figure 3.4 illustrates this example, and also shows several level curves of the function f , i.e., $f(x, y) = 1$ (panel (a)), $f(x, y) = 3$ (panel (b)) and $f(x, y) = 2$ (panel (c)).

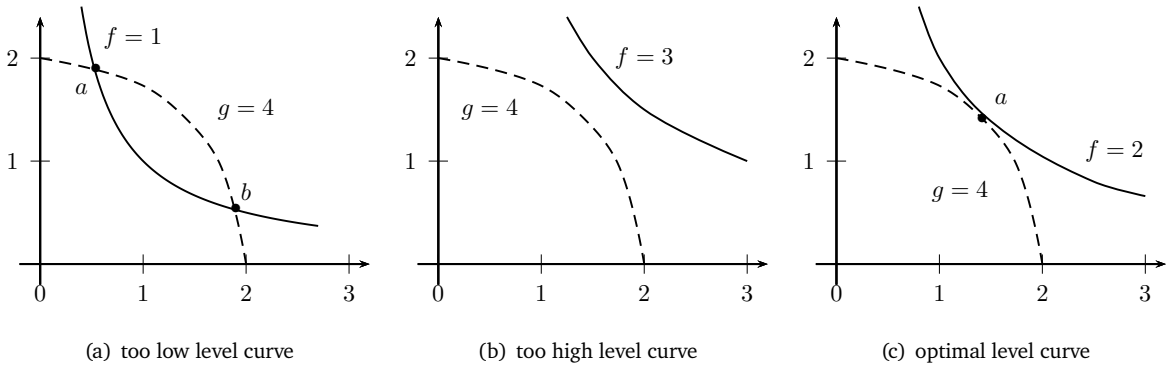


FIGURE 3.4 – Illustration of Problem (3.2).

The dashed segment of the circle represents all points that satisfy the equality constraint. In other words, it denotes the set of points from which the optima of the objective function f can be determined, that is, the feasible region.

The solid curve in Figure 3.4(a) depicts all points in the domain of f for which the function f attains the value one. Both points a and b lie within the feasible region and, in addition, are located on the level curve of f corresponding to this value. Figure 3.4(b) illustrates an infeasible case. We observe that the level curve of f - that is, the set of points where f has a value three - does not intersect the feasible region. In other words, a value of three units cannot be attained by an appropriate choice of (x, y) in the feasible set. Finally, Figure 3.4(c) depicts the optimal scenario, where the function f reaches a value of two at point a . If a value greater than two were targeted, the situation shown in panel (b) would occur. Hence, given the feasible region, f cannot attain a higher value.

In Figure 3.5, we visualized the optimal situation in some more detail.

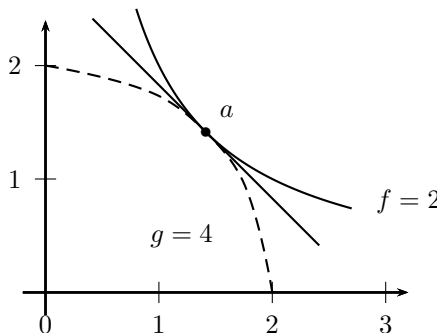


FIGURE 3.5 – Optimal allocation in Problem 3.2.

In the optimal situation, the tangent line to the graph of the feasible region coincides with the tangent line to the level curve of f . That is, both tangent lines share the same slope. This naturally raises the

question of whether these slopes can be calculated. As we will see shortly, they can. To explore this, let us revisit the feasible region in Figure 3.5. From the graph, we observe that the feasible region can be described by a function $y = h(x)$, meaning that all points within this region satisfy the relationship

$$G(x) = g(x, h(x)) = b.$$

That is, $G(x)$ is a constant function. Therefore, the derivative of $G(x)$ is zero for all x . However, because $G(x) = g(x, h(x))$, the derivative of $G(x)$ is also equal to

$$G'(x) = D_1g(x, h(x)) + D_2g(x, h(x))h'(x).$$

Combining the two expressions for $G'(x)$ then leads to the conclusion that, apparently,

$$D_1g(x, h(x)) + D_2g(x, h(x))h'(x) = 0.$$

But, this implies that

$$h'(x) = -\frac{D_1g(x, h(x))}{D_2g(x, h(x))}.$$

In other words, we have derived an explicit formula for the slope of the tangent line to the graph of the feasible region at any point.

Next, consider the level curve of f corresponding to a value of k . Again, as illustrated in Figure 3.5, all points on this level curve lie on the graph of some function $y = h_2(x)$. By applying the same reasoning as before, we find that the derivative of this function is equal to

$$h_2'(x) = -\frac{D_1f(x, h_2(x))}{D_2f(x, h_2(x))}.$$

Since the slopes coincide at the optimal point, we conclude that a necessary condition for the optimization problem (3.2) to have a solution is that the following equality is satisfied:

$$-\frac{D_1g(x, h(x))}{D_2g(x, h(x))} = -\frac{D_1f(x, h_2(x))}{D_2f(x, h_2(x))}.$$

Or, put differently, at an optimum (x_0, y_0) next two conditions hold:

$$1) \quad g(x_0, y_0) = b \tag{3.3}$$

$$2) \quad \frac{D_1g(x_0, y_0)}{D_2g(x_0, y_0)} = \frac{D_1f(x_0, y_0)}{D_2f(x_0, y_0)}. \tag{3.4}$$

Condition (3.4) is called the *first-order criterion for a constrained optimum*.

EXAMPLE 3.6 (*optimal use of production factors*)

Consider a producer aiming to sell a fixed quantity of goods $\bar{y}$ on the market. Suppose his production function is $Q(K, L)$, where K and L represent the amounts of capital and labor used in production, respectively, and that he can freely choose the levels of capital and labor. Let the cost per unit of capital be r and the cost per unit of labor be ω . To minimize his total cost, the producer faces the following optimization problem:

$$\text{minimize } C(K, L) = rK + \omega L \text{ subject to } Q(K, L) = \bar{y}.$$

From the first-order criterion (3.4), we conclude that at the point of minimum we have

$$\frac{C'_K}{C'_L} = \frac{Q'_K}{Q'_L} \text{ or } \frac{r}{\omega} = \frac{Q'_K}{Q'_L}.$$

Thus, if, for example, the production function is given by the Cobb-Douglas form $Q(K, L) = K^\alpha L^{1-\alpha}$, it follows that at the point minimizing production costs we have

$$\frac{r}{\omega} = \frac{\alpha K^{\alpha-1} L^{1-\alpha}}{(1-\alpha) K^\alpha L^{-\alpha}} \text{ and } K^\alpha L^{1-\alpha} = \bar{y}.$$

A straightforward rearrangement shows that these conditions can be expressed as

$$\frac{r}{\omega} = \frac{L}{K} \text{ and } K^\alpha L^{1-\alpha} = \bar{y}.$$

From the first equality, it follows that $K = \frac{\omega}{r} L$. Substituting this into the production function shows that at the cost-minimizing point, $L = \left(\frac{r}{\omega}\right)^\alpha \bar{y}$. The corresponding amount of capital is $K = \left(\frac{r}{\omega}\right)^{\alpha-1} \bar{y}$. Hence, the minimum cost is: $C(K, L) = r \left(\frac{r}{\omega}\right)^{\alpha-1} \bar{y} + \omega \left(\frac{r}{\omega}\right)^\alpha \bar{y} = 2r^\alpha \omega^{1-\alpha} \bar{y}$.

EXERCISE 3.3 Solve the next constrained optimization problems.

- Maximize $f(K, L) = KL$ subject to $K + L = 36$ (you may assume a maximum exists).
- Optimize $f(x, y) = 2x + 3y$ subject to $x^2 + xy + y^2 = 21$.
- Maximize $f(x, y) = x^{\frac{2}{3}} y^{\frac{1}{3}}$ ($x > 0$, $y > 0$) subject to $x + 2y = 10$ (you may assume a maximum exists).
- Optimize $f(x, y) = x^2 + y^2$ subject to $x^2 + xy + y^2 = 3$.

To derive the first-order condition (3.4), we assumed that, for both the "budget constraint", $g(x, y) = b$ and the "level curve", $f(x, y) = k$, the variable y could be expressed as a differentiable function of x . In most cases, this poses no difficulty, but there are situations where it is not possible. Figure 3.6 provides an example of such a case. In this figure, we plotted the constraint set $g(x, y) = x^2 + y^2 = 1$. From the figure, we can see that at almost every point on this set, y can locally be expressed as a function of x . In other words, for nearly every point $(\bar{x}, \bar{y})$ on the constraint set, there exists a small

neighborhood (a "box") in which the constraint set can be represented as a function $y = h(x)$. There are, however, two exceptions: the points $(1, 0)$ and $(-1, 0)$. At these locations, it is impossible to define such a neighborhood where the constraint set can be viewed as a differentiable function $y = h(x)$. Consider, e.g., $(1, 0)$. For every x within the interval (x_l, x_r) , we observe that there are - except at $x = 1$ - either two corresponding points on the graph or none. Panel (b) of Figure 3.6 provides a zoomed-in view of this part of the constraint set.

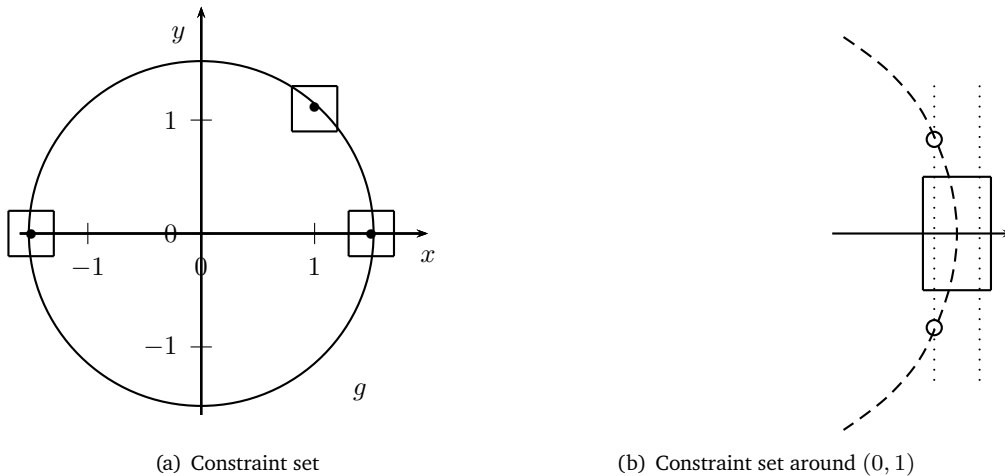


FIGURE 3.6 – Constraint set $g(x, y) = x^2 + y^2 = 1$

Note that near the points $(1, 0)$ and $(-1, 0)$, the constraint set can instead be expressed as a function of y . In other words, as shown in the box in panel (b) of Figure 3.6, the constraint set can be represented by a differentiable function $x = h(y)$. Therefore, for the constraint set $g(x, y) = x^2 + y^2 = 1$ we can always find, locally, either a differentiable function $y = h(x)$ or a differentiable function $x = h(y)$ that describes the set.

Unfortunately, this does not hold for all constraint sets. Consider the constraint set given by $g(x, y) = x^2 - y^2 = 0$. We plotted this set in Figure 3.7.

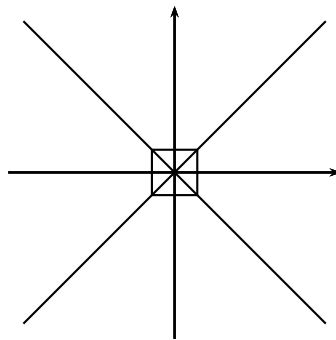


FIGURE 3.7 – Constraint set $g(x, y) = x^2 - y^2 = 0$.

Clearly, at $(0, 0)$ the constraint set can neither be described as a differentiable function $y = h(x)$ or as a differentiable function $x = h(y)$. This implies, that the above sketched procedure to arrive at the first order condition for a constrained optimum (3.4) only works under some additional assumptions. That is, assumptions ensuring that, locally, the constraint set/level curve can be represented either as a differentiable function $y = h(x)$ or as a differentiable function $x = h(y)$. The following theorem, known as the Implicit Function Theorem, establishes sufficient conditions under which this result follows.

THEOREM 3.7 (*Implicit Function Theorem*)

Let $g(x, y)$ be a continuously differentiable function in a ball around (c, d) . Suppose $g(c, d) = b$. Then, if

$$D_2g(c, d) \neq 0,$$

there exists on an interval I around $x = c$ a differentiable function h such that $y = h(x)$ is the solution of $g(x, y) = b$. Moreover, $h'(x) = -\frac{D_1g(x, y)}{D_2g(x, y)}$.

In case

$$D_1g(c, d) \neq 0,$$

there exists on an interval I around $y = d$ a differentiable function h such that $x = h(y)$ is the solution of $g(x, y) = b$. Moreover, $h'(y) = -\frac{D_2g(x, y)}{D_1g(x, y)}$.

In words, the theorem states that if a function $g(x, y)$ is continuously differentiable in a neighborhood of a point (c, d) , and if the partial derivative of g with respect to y is nonzero at (c, d) , then the following holds. There exists an interval I around $x = c$ such that for every $x_0 \in I$ there is a unique corresponding y_0 satisfying the equation $g(x, y) = b$ (where $g(c, d) = b$). Hence, the relation $h(x_0) := y_0$ for $x_0 \in I$ defines a function on I which is, moreover, differentiable. The derivative of h can be computed explicitly using the formula given in the theorem above.

A similar conclusion can be drawn in case the partial derivative of g with respect to x differs from zero at (c, d) . Consequently, if $g'(c, d) \neq [0 \ 0]$, there exists a neighborhood of (c, d) containing points (x_0, y_0) that also satisfy $g(x, y) = b$. Moreover, this constraint set can locally be represented either as a differentiable function of x around $x = c$, or as a differentiable function of y , around $y = d$.

As a consequence, if at every point (x_0, y_0) of the constraint set $g(x, y) = b$ (with $g \in C^1$) we have $g'(x_0, y_0) \neq [0 \ 0]$, then the constraint set can everywhere be viewed locally either as a differentiable function of x or as a differentiable function of y .

EXAMPLE 3.8

1. Consider the constraint set $g(x, y) = x^2 + y^2 = 1$. Then $g'(x, y) = [2x \ 2y]$. So, $g \in C^1$. Furthermore, both partial derivatives of g are simultaneously zero only in case $(x, y) = (0, 0)$. However, $(0, 0)$ is not a point from the constraint set, i.e., $g(0, 0) = 0 \neq 1$. So, by Theorem 3.7, at every point

of our constraint set we can find a differentiable function that describes the constraint set in a neighborhood of that point.

2. Consider the constraint set $g(x, y) = x^2 - y^2 = 0$. Then $g'(x, y) = [2x \ -2y]$. So, $g \in C^1$. Furthermore, both partial derivatives of g are simultaneously zero only in case $(x, y) = (0, 0)$. So, by Theorem 3.7, at every point of our constraint set, except $(0, 0)$, we can find a differentiable function that describes the constraint set in a neighborhood of that point. Notice the theorem does not claim that at $(0, 0)$ such a differentiable function does not exist.
3. Consider the constraint set $g(x, y) = (x - y)^2 = 0$. Then $g'(x, y) = 2(x - y)[1 \ -1]$. So, $g \in C^1$. Furthermore, both partial derivatives of g are simultaneously zero at all points of the constraint set. So, Theorem 3.7, is not applicable at any point of the constraint set. However, it is clear that we can define a differentiable function that describes the constraint set in a neighborhood of any point on the constraint set; namely, $y = f(x) := x$.
4. Consider the constraint set $g(x, y) = |x| - y = 0$. Then, for all points (x, y) with $x > 0$, $g'(x, y) = [1 \ -1]$, while for all points with $x < 0$, it is $g'(x, y) = [-1 \ -1]$. Hence, $g \in C^1$ and $g'(x, y) \neq [0 \ 0]$ for all $(x, y) \neq (0, 0)$.

Consequently, by Theorem 3.7, at every point $(x, y) \neq (0, 0)$ on the constraint set, there exists a differentiable function that locally represents the constraint set in a neighborhood of that point.

Finally, note that g is not differentiable at $(0, 0)$. Thus, Theorem 3.7 does not yield a definitive conclusion as to whether, at the point $(0, 0)$, the constraint set allows x to be expressed as a function of y , y to be expressed as a function of x , or whether neither representation is possible.

EXERCISE 3.4 Determine for next constrained sets $S := \{(x, y) \mid f(x, y) = b\}$ all points (x_0, y_0) where in an interval around x_0 there exists a continuously differentiable function $y = h(x)$ such that, according the implicit function theorem, $g(x, h(x)) = b$. Calculate $h'(x_0)$ at those points.

- a. $S = \{(x, y) \mid (x - y)^2 = 4\}$.
- b. $S = \{(x, y) \mid (x - 1)^2 + (y - 2)^2 = 4\}$.

Using the Implicit Function Theorem, we can now formulate a formal first-order condition for our constrained optimization problem (3.2).

THEOREM 3.9 (*first-order criterion for a constrained optimum*)

Consider the constrained optimization problem (3.2). Assume both $f, g \in C^1$. Assume the problem has a solution at (c, d) . Then, if both $D_2f(c, d)$ and $D_2g(c, d)$ differ from zero,

$$1) \ g(c, d) = b \text{ and } 2) \ \frac{D_1g(c, d)}{D_2g(c, d)} = \frac{D_1f(c, d)}{D_2f(c, d)}.$$

EXERCISE 3.5 Determine, using Theorem 3.9, all points (x, y) where an extremum may occur for next constrained optimization problems.

- Optimize $x^2 + y^2$ subject to $xy = 1$.
- Optimize $e^{x^2+y^2}$ subject to $x^4 + y^4 = 2$.

In fact, using the Implicit Function Theorem, one can derive an alternative first-order condition under somewhat less restrictive assumptions. This approach is known as Lagrange's method. Before presenting its general formulation, we first outline the reasoning that leads to this method.

Consider again our constrained optimization problem (3.1) and suppose it has a solution at the point (c, d) . Then $g(c, d) = b$. Assume, for example, that in a neighborhood of $x = c$ the constraint set can be described by a function $y = h(x)$. That is,

$$G(x) := g(x, h(x)) = b$$

for all x in some interval I around $x = c$. Moreover, on this part of the constraint set the objective function $f(x, y)$ reduces to the function

$$F(x) := f(x, h(x)), \quad x \in I.$$

Since $G(x) = b$ on I , it follows that $G'(x) = 0$, for all $x \in I$. Thus, applying the chain rule of differentiation,

$$D_1g(x, h(x)) + D_2g(x, h(x))h'(x) = 0 \text{ for all } x \in I. \quad (3.5)$$

Moreover, because f is assumed to attain an optimum at (c, d) , the function $F(x)$ attains an optimum at $x = c$. Hence, $F'(c) = 0$. Observe that

$$F'(x) = D_1f(x, h(x)) + D_2f(x, h(x))h'(x).$$

so that $F'(c) = D_1f(c, d) + D_2f(c, d)h'(c)$. From equation (3.5) it follows that $h'(c) = -\frac{D_1g(c, d)}{D_2g(c, d)}$. Substituting $h'(c)$ into $F'(c)$ gives

$$D_1f(c, d) - \frac{D_2f(c, d)}{D_2g(c, d)}D_1g(c, d) = 0.$$

Introducing $\lambda := \frac{D_2f(c, d)}{D_2g(c, d)}$, the above equation can be rewritten as

$$D_1f(c, d) - \lambda D_1g(c, d) = 0.$$

Furthermore, the defining equation of $\lambda := \frac{D_2f(c, d)}{D_2g(c, d)}$ can be rewritten as follows:

$$D_2f(c, d) - \lambda D_2g(c, d) = 0.$$

Thus, if the problem has an optimum at (c, d) , the last two equations guarantee the existence of a scalar λ such that:

$$f'(c, d) - \lambda g'(c, d) = [0 \ 0].$$

With this, we proved next result.

THEOREM 3.10 *Consider the constrained optimization problem (3.2). Assume both $f, g \in C^1$. Assume the problem has a solution at (c, d) . Then, if $g'(c, d) \neq [0 \ 0]$, there exists a $\lambda \in \mathbb{R}$ such, that*

$$f'(c, d) - \lambda g'(c, d) = [0 \ 0].$$

That is, the gradient of the objective function is a multiple of the constraint function.

The scalar λ is referred to as the *Lagrange multiplier*. By defining an auxiliary function $L(x, y, \lambda)$, the first-order conditions of Lagrange's theorem can be derived in a straightforward manner.

DEFINITION 3.11 The function L , defined by

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b),$$

is called the *Lagrange function* of the optimization problem (3.2).

Using this definition, Lagrange's theorem can be restated as follows.

THEOREM 3.12 (*Lagrange's Theorem*).

Consider the constrained optimization problem (3.2). Assume both $f, g \in C^1$. Assume the problem has a solution at (c, d) . Then, if $g'(c, d) \neq [0 \ 0]$, there exists a $\lambda \in \mathbb{R}$ such, that (c, d, λ) is a stationary point of the Lagrange function.

Theorem 3.12 leads us to the following algorithm for solving the constrained optimization problem (3.2).

ALGORITHM 3.13 To find the candidate solutions of our constrained optimization problem (3.2) proceed as follows:

Step 1. Verify whether $f, g \in C^1$.

Step 2. Find all points where $g' = [0 \ 0]$ and $g(x, y) = b$.

Step 3. Write down the Lagrange function

$$L(x, y, \lambda) := f(x, y) - \lambda(g(x, y) - b).$$

Step 4. Determine all solutions (x, y, λ) of $L'(x, y, \lambda) = [0 \ 0 \ 0]$.

All points found in Step 2 and Step 4, above, are candidate solutions for the constrained optimization problem (3.2).

Step 2 in the above algorithm is included because at points in the feasible region where the derivative is zero, the Implicit Function Theorem no longer applies. Since our theory relies on the assumption that this theorem holds, we cannot make conclusions at points where it fails. In particular, the function might attain an optimum at such points.

Moreover, if the functions f and g are not continuously differentiable, the entire procedure outlined above breaks down. In that case, alternative methods must be employed to solve the problem.

EXAMPLE 3.14 Consider the constrained optimization problem

$$\text{optimize } f(x, y) = 4x - 3y \text{ subject to } g(x, y) = x^2 + y^2 = 25.$$

First note that, our objective function is continuous and the constraint set is a compact set. So, by Weierstrass' theorem, this optimization problem has both a global maximum and a global minimum. To find these optima, we proceed along the lines of Algorithm 3.13.

Step 1. $f' = [4 \ -3]$ and $g' = [2x \ 2y]$. Clearly, all partial derivatives are continuous. So, $f, g \in C^1$.

Step 2. $g'(x, y) = [0 \ 0]$ only in case $(x, y) = (0, 0)$. However, $g(0, 0) = 0 \neq 25$. So, $(0, 0)$ is not in the feasible region. We, therefore, don't find any candidate solutions in this step.

Step 3. The Lagrange function is $L(x, y, \lambda) = 4x - 3y - \lambda(x^2 + y^2 - 25)$.

Step 4. Differentiation of the Lagrange function shows that the candidate solutions must satisfy

$$L' = [4 - 2\lambda x \ -3 - 2\lambda y \ x^2 + y^2 - 25] = [0 \ 0 \ 0].$$

Note, $\lambda = 0$ does not provide a solution. From $L'_x = 0$ and $L'_y = 0$ we have $x = \frac{2}{\lambda}$ and $y = -\frac{3}{2\lambda}$, respectively. Substitution of this into the constraint $x^2 + y^2 = 25$ yields $(\frac{2}{\lambda})^2 + (\frac{-3}{2\lambda})^2 = 25$. Some simple rewriting of this last equation shows next that $25 = 100\lambda^2$. So, $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$ solve the equation. This yields the two candidate solutions $(x_1, y_1, \lambda_1) = (4, -3, \frac{1}{2})$ and $(x_2, y_2, \lambda_2) = (-4, 3, -\frac{1}{2})$.

Note that $f(x_1, y_1) = 25 > -25 = f(x_2, y_2)$. Since there are only two candidate solutions we conclude, by Weierstrass' theorem, that f has a global maximum at $(4, -3)$ and a global minimum at $(-4, 3)$.

EXAMPLE 3.15 (*Markowitz Mean-Variance Optimal Portfolio*)

This example illustrates how to determine the optimal portfolio for an investor who wishes to achieve a predetermined return while minimizing the risk of his capital X . Assume this investor can either invest his money on a savings account, where he will get an almost sure interest rate of $r\%$, he can invest in bonds, where he may expect a return of $r_b\%$ (with $r_b > r$), or he can invest his money in

stocks, where his investment has the highest expected return of $r_s\%$. Of course, investing his money in bonds and/or stocks implies that there is a higher uncertainty whether this return will be realized. So, the investor has to make a decision on which fraction ω_b from his capital X he should invest in bonds, the fraction ω_s he should invest in stocks and the remaining fraction $(1 - \omega_b - \omega_s)$ he should put on his savings account. Assume he is looking for a portfolio which provides him an expected return of $\bar{X}$ (where, of course, this return is bounded by the minimum achievable and maximum achievable return, i.e., $(1 + r)X \leq \bar{X} \leq (1 + r_s)X$). Then it seems reasonable to assume that the investor will look for that portfolio which will almost sure provide him this return $\bar{X}$. Under the assumption that the returns on the savings account, bonds and stocks are normally distributed with a variance of σ_0^2 , σ_b^2 and σ_s^2 , respectively, one can show that the return of a portfolio where a fraction ω_b is invested in bonds, ω_s is invested in shares and the rest put on a savings account will be normally distributed too, with an expected return of $\bar{X}$ and a variance σ^2 . Clearly, the smaller this variance is, the higher the probability is that this expected return of $\bar{X}$ will be realized. For that reason Markowitz ([14]) proposed to choose, from all portfolios that provide an expected return of $\bar{X}$, that portfolio which has a minimum variance. The variance of the portfolio is $\sigma^2 = [1 - \omega_b - \omega_s \ \omega_b \ \omega_s] \Sigma [1 - \omega_b - \omega_s \ \omega_b \ \omega_s]^T$, where Σ is the covariance matrix for the returns on the assets in the portfolio. For simplicity assume that these returns are not correlated, so that Σ is the diagonal matrix with entries σ_i^2 , $i = 0, b, s$, respectively. So, the portfolio that has minimum "risk" in realizing the desired return of $\bar{X}$ is obtained by solving the next constrained optimization problem.

$$\begin{aligned} & \text{minimize}_{\omega_b, \omega_s} \sigma^2(\omega_b, \omega_s) := (1 - \omega_b - \omega_s)^2 \sigma_0^2 + \omega_b^2 \sigma_b^2 + \omega_s^2 \sigma_s^2 \\ & \text{subject to } \mu(\omega_b, \omega_s) := (1 - \omega_b - \omega_s)(1 + r)X + \omega_b(1 + r_b)X + \omega_s(1 + r_s)X = \bar{X}. \end{aligned}$$

Clearly, $\sigma^2(\omega_b, \omega_s)$ is continuous and we have to find the minimum of this function on a compact set. So, by Weierstrass' theorem, there exists an optimal portfolio. We use Algorithm 3.13 again to find the minimum. To that end, first note, that both σ^2 and μ are continuous differentiable. It is also directly verified that Step 2 in the algorithm does not provide any candidate solutions. Therefore, we immediately proceed with considering the Lagrange function:

$$L(\omega_b, \omega_s, \lambda) := \sigma^2(\omega_b, \omega_s) - \lambda(\mu(\omega_b, \omega_s) - \bar{X}).$$

Straightforward differentiation shows that

$$\begin{aligned} L'_{\omega_b} &= -2(1 - \omega_b - \omega_s)\sigma_0^2 + 2\omega_b\sigma_b^2 - \lambda(-(1 + r)X + (1 + r_b)X) \\ L'_{\omega_s} &= -2(1 - \omega_b - \omega_s)\sigma_0^2 + 2\omega_s\sigma_s^2 - \lambda(-(1 + r)X + (1 + r_s)X) \\ L'_\lambda &= (1 - \omega_b - \omega_s)(1 + r)X + \omega_b(1 + r_b)X + \omega_s(1 + r_s)X - \bar{X}. \end{aligned}$$

Some simple rewriting shows that, with $x := [\omega_b \ \omega_s \ \lambda]^T$, the equation $L' = [0 \ 0 \ 0]$ can be rewritten as the matrix equation $Ax = b$, where

$$A = \begin{bmatrix} 2(\sigma_0^2 + \sigma_b^2) & 2\sigma_0^2 & (r - r_b)X \\ 2\sigma_0^2 & 2(\sigma_0^2 + \sigma_s^2) & (r - r_s)X \\ (r_b - r)X & (r_s - r)X & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2\sigma_0^2 \\ 2\sigma_0^2 \\ \bar{X} - (1 + r)X \end{bmatrix}.$$

To determine the solution of this matrix equation, we have to find the inverse of matrix A . One can verify (e.g., by verifying that AA^{-1} equals the identity matrix) that $\det * A^{-1}$ equals

$$\begin{bmatrix} X^2(r - r_s)^2 & -X^2(r - r_b)(r - r_s) & 2X((r_b - r_s)\sigma_0^2 + (r_b - r)\sigma_s^2) \\ -X^2(r - r_b)(r - r_s) & X^2(r - r_b)^2 & -2X((r_b - r_s)\sigma_0^2 + (r - r_s)\sigma_b^2) \\ -2X((r_b - r_s)\sigma_0^2 + (r_b - r)\sigma_s^2) & 2X((r_b - r_s)\sigma_0^2 + (r - r_s)\sigma_b^2) & 4(\sigma_0^2\sigma_b^2 + \sigma_0^2\sigma_s^2 + \sigma_b^2\sigma_s^2) \end{bmatrix},$$

where $\det$, the determinant of A , equals $2X^2((r_b - r_s)^2\sigma_0^2 + (r - r_s)^2\sigma_b^2 + (r - r_b)^2\sigma_s^2)$.

By calculating next $x = A^{-1}b$ we obtain the optimal parameters for the portfolio

$$\begin{aligned} \omega_b &= \frac{2X}{\det} (X\sigma_0^2(r_s - r)(r_s + r_b) + ((r_b - r_s)\sigma_0^2 + (r_b - r)\sigma_s^2)(\bar{X} - (1 + r)X)) \\ \omega_s &= \frac{2X}{\det} (X\sigma_0^2(r_b - r)(r_s + r_b) - ((r_b - r_s)\sigma_0^2 + (r - r_s)\sigma_b^2)(\bar{X} - (1 + r)X)), \end{aligned}$$

with a corresponding λ given by

$$\lambda = \frac{2}{\det} (X\sigma_0^2((r - r_b)\sigma_s^2 + (r - r_s)\sigma_b^2) + 2(\sigma_0^2\sigma_b^2 + \sigma_0^2\sigma_s^2 + \sigma_b^2\sigma_s^2)(\bar{X} - (1 + r)X)).$$

EXERCISE 3.6 Use Lagrange's method to solve the next constrained optimization problems.

- Optimize $f(x, y) = xy$ subject to $4x^2 + y^2 = 32$.
- Optimize $f(x, y) = x + y$ subject to $x^2 + xy + y^2 = 3$.
- Optimize $Q(K, L) = K + 20L$ subject to $\sqrt{K} + L = 30$, $K \geq 0$, $L \geq 0$.
- Maximize $f(x, y) = x$ subject to $x^3 + y^2 = 0$.

EXERCISE 3.7 Use Lagrange's method to solve the constrained optimization problems from Exercise 3.3.

EXERCISE 3.8 A consumer has a utility function $U(x, y) = bx^\alpha y^{1-\alpha}$, where x and y are the amount of product 1 and 2 the consumer can choose from, respectively. Assume the price of product 1 is p_1 , the price of product 2 is p_2 and his income is I .

- Assume the consumer wants to maximize his utility. Formulate the corresponding constrained optimization problem.
- Determine the solution of this constrained optimization problem.

- c. Calculate the approximate change in the demand for good x and utility U if the income, I , is changed by one unit.

3.3 The Envelope Theorem and Lagrange Multiplier

In this subsection, we examine in more detail the role of the budget b in the constraint set $g(x, y) = b$. We will show that increasing the budget b by one unit approximately increases the optimal value of the objective function by an amount equal to λ , the corresponding Lagrange multiplier.

To show this, we first consider an unconstrained optimization problem that includes a parameter. We will analyze how the optimal value of the objective function varies as this parameter changes. For simplicity, we focus on an objective function of two variables with a single parameter. The general case (n variables and m parameters) can be dealt with analogously.

So, consider the optimization of

$$f(x, y, a) \text{ with respect to } x \text{ and } y, \quad (3.6)$$

where a is some parameter and $f \in C^2$.

The following example illustrates the motivation for the problem setting.

EXAMPLE 3.16 Consider the maximization of profit by a producer of a good in two separate markets. His price, p_i , he receives in both markets depends on the amount he supplies. For the first market this price equals $d_1 - x$ and in the second market it is $d_2 - y$. Where x is the amount of the good he supplies to market 1, y is the amount he supplies to market 2 and d_i is the demand for the good in market i , $i = 1, 2$. Assume that his marginal production cost are c and his fixed cost one. The maximization problem boils down then to the maximization of the next function $\pi(x, y)$ that contains three parameters: d_1 , d_2 and c , respectively.

$$\pi(x, y) = x(d_1 - x) + y(d_2 - y) - c(x + y) - 1.$$

The producer might then be interested in how his optimal supply in the two markets, responds to these parameters.

Assuming that Problem (3.6) is solved at (x_0, y_0) for a fixed value $a = a_0$, the first-order conditions imply that

$$D_1 f(x_0, y_0, a_0) = 0 \text{ and } D_2 f(x_0, y_0, a_0) = 0. \quad (3.7)$$

Using a generalized version of the Implicit Function Theorem (see Theorem 3.26, Section 3.5), the pair of equations can be locally solved for (x, y) as differentiable functions $X_0(a)$ and $Y_0(a)$ of a near

a_0 and (x_0, y_0)), provided that the derivative matrix

$$D^2 f := \begin{bmatrix} D_{11}f(x, y, a) & D_{12}f(x, y, a) \\ D_{21}f(x, y, a) & D_{22}f(x, y, a) \end{bmatrix},$$

is invertible at (x_0, y_0, a_0) .

Assuming that this invertibility condition is satisfied, we now examine how the objective function responds to a change in the parameter a , at $a = a_0$. This response is given by $\frac{d(f(X_0(a), Y_0(a), a))}{da}$. Using the chain rule of differentiation we have

$$\begin{aligned} \frac{d(f(X_0(a), Y_0(a), a))}{da} = \\ D_1 f(X_0(a), Y_0(a), a) X_0'(a) + D_2 f(X_0(a), Y_0(a), a) Y_0'(a) + D_3 f(X_0(a), Y_0(a), a). \end{aligned}$$

Thus, by applying the first-order conditions (3.7), the response at $a = a_0$ is given by

$$\frac{d(f(X_0(a_0), Y_0(a_0), a_0))}{da} = 0 + 0 + D_3 f(X_0(a), Y_0(a), a).$$

Hence, we showed the next result.

THEOREM 3.17 (*Envelope Theorem*)

Assume $f(x, y, a) \in C^2$ and f has for $a = a_0$ a stationary point at (x_0, y_0) . Then, if $D^2 f(x_0, y_0, a_0)$ is invertible, f has a stationary point $(X_0(a), Y_0(a))$ in a neighborhood of $a = a_0$. Furthermore, both $X_0(a)$ and $Y_0(a)$ are differentiable functions of a around $a = a_0$. Moreover,

$$\frac{df(X_0(a), Y_0(a), a)}{da} = D_3 f(X_0(a), Y_0(a), a).$$

EXAMPLE 3.18 Reconsider Example 3.16. Consider the case that the parameters have the next values: $d_1 = 60$, $d_2 = 80$ and $c = 20$. Then a maximum profit of 1275 units is attained by selling $x_0 = 15$ units and $y_0 = 30$ units in market 1 and 2, respectively. Next assume that the marginal cost, c , is increased by one unit.

Note, that $\pi'(x, y, c) = [d_1 - 2x - c \quad d_2 - 2y - c \quad -x - y]$ and, therefore,

$$\pi'' = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

All the entries of this matrix are continuous, so, $\pi \in C^2$. Furthermore, at any point (x_0, y_0, c) ,

$$D^2\pi := \begin{bmatrix} D_{11}\pi(x, y, c) & D_{12}\pi(x, y, c) \\ D_{21}\pi(x, y, c) & D_{22}\pi(x, y, c) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Clearly, $D^2\pi$ is invertible. From the Envelope Theorem we conclude then that

$$\frac{d\pi(X_0(c), Y_0(c), c)}{dc} = D_3\pi(x, y, c) = -(x + y).$$

So, an increase of one unit in the marginal cost leads to a reduction in the profit of, approximately, $15 + 30 = 45$ units.

EXERCISE 3.9

- What is the effect if the parameter $a = a_0$ ($a > 0$) is increased by one unit on the maximum value of $f(x, a) = -x^4 + ax + a^3$.
- What is the effect if the parameter $a = a_0$ ($a > 0$) is increased by one unit on the maximum value of $f(x, y, a) = -x^2 - y^4 + ax + a^2y^2 + 4$.

EXERCISE 3.10 (profit maximization by a firm)

Consider a firm producing a good. The cost of the firm for producing x units of the good is $C(x)$. Assume that this cost function has the next monotonicity properties: $C'(x) > 0$ and $C''(x) > 0$. Assume that a fraction α of the amount of produced product is sold on the market at a price p per unit. Show there is a unique amount of goods where profit is maximized. Calculate by how much maximum profit increases if α increases.

Next, reconsider the constrained optimization problem

$$\text{optimize } f(x, y) \text{ subject to } g(x, y) = b. \quad (3.8)$$

Assume that this constrained optimization problem has a solution at (x_0, y_0) . By Lagrange's theorem, and under appropriate regularity conditions, this solution (x_0, y_0) can be obtained by considering the corresponding Lagrange function

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b).$$

Now, to evaluate the effect of the 'budget' b on the optimal performance - which coincides with the value of $L(x, y, \lambda)$ on the constrained set - we treat b as a parameter in the optimization problem above. If the function $\tilde{L}(x, y, \lambda, b) := f(x, y) - \lambda(g(x, y) - b)$ satisfies the conditions of the Envelope

Theorem, we may conclude that

$$\frac{d\tilde{L}(X_0(b), Y_0(b), \Lambda_0(b), b)}{db} = D_4 \tilde{L} = \lambda.$$

Or, stated differently, if the constrained optimization problem has a solution at (x_0, y_0) and (x_0, y_0, λ_0) is a stationary point of the Lagrange function with $f(x_0, y_0) = \bar{f}$, then an increase in the budget from b to $b + \Delta b$ leads to an approximate increase of the objective value at the new optimum to $\bar{f} + \lambda_0 \Delta b$. This provides the motivation for the following interpretation of the Lagrange multiplier in a constrained optimization problem.

COROLLARY 3.19 (*Shadow Price Interpretation Lagrange multiplier*)

Consider the constrained optimization problem (3.8) with $f, g \in C^2$. Assume the problem has a solution at (c, d) and $g'(c, d) \neq [0 \ 0]$. Let (c, d, λ_0) be a stationary point of the corresponding Lagrange function. Then, if matrix

$$\begin{bmatrix} Hf - \lambda^* Hg & -(Dg)^T \\ -Dg & 0 \end{bmatrix}$$

is invertible at (c, d, λ_0) :

$$\frac{df(x, y)}{db} = \lambda_0 \text{ at } (x, y) = (c, d).$$

In economics, Corollary 3.19 is typically interpreted as stating that the shadow price, λ_0 , measures how rapidly the optimal objective value improves when the constraint is relaxed.

EXAMPLE 3.20 Reconsider Example 3.14. In this example, we showed that the constrained optimization problem

$$\text{maximize } f(x, y) = 4x - 3y \text{ subject to } g(x, y) = x^2 + y^2 = 25$$

has a global maximum at $(x_1, y_1) = (4, -3)$, with corresponding Lagrange multiplier $\lambda_0 = \frac{1}{2}$. Note that $g'(4, -3) = [8 \ -6] \neq [0 \ 0]$ and

$$\begin{bmatrix} Hf - \lambda_0 Hg & -(Dg)^T \\ -Dg & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -8 \\ 0 & -1 & 6 \\ -8 & 6 & 0 \end{bmatrix}$$

is invertible. Therefore, by Corollary 3.19, if the "budget" 25 increases by $\frac{1}{4}$, the maximum of the adjusted constrained optimization problem will approximately increase by $\frac{1}{2} * \frac{1}{4} = \frac{1}{8}$.

A similar analysis shows that, at the global minimum $(x_2, y_2) = (-4, 3)$, the Lagrange multiplier is $-\frac{1}{2}$. Hence, if the "budget" increases by $\frac{1}{4}$, the minimum will approximately change to $-25 - \frac{1}{8}$.

EXAMPLE 3.21 Reconsider Example 3.15. In this example we showed that the Lagrange parameter corresponding with the optimal portfolio is

$$\lambda = \frac{2}{\det} (X\sigma_0^2((r - r_b)\sigma_s^2 + (r - r_s)\sigma_b^2) + 2(\sigma_0^2\sigma_b^2 + \sigma_0^2\sigma_s^2 + \sigma_b^2\sigma_s^2)(\bar{X} - (1 + r)X)).$$

According to Corollary 3.19, this parameter is given by

$$\frac{d\sigma^2(\omega_b, \omega_s)}{d\bar{X}}.$$

By computing this parameter, the investor obtains an estimate of the approximate effect on risk if (s)he wishes to target a different expected return $\bar{X}$. For example, if the investor increases $\bar{X}$ by one unit, the corresponding risk σ^2 will approximately change by λ .

EXERCISE 3.11 Reconsider items a), b) and c) from Exercise 3.6. Use Lagrange's method to solve the next constrained optimization problems. Compare the optimum value with the value predicted by Corollary 3.19, using the Lagrange parameter you calculated in Exercise 3.6 for the corresponding item.

- a. Optimize $f(x, y) = xy$ subject to $4x^2 + y^2 = 32.01$.
- b. Optimize $f(x, y) = x + y$ subject to $x^2 + xy + y^2 = 4$.
- c. Optimize $Q(K, L) = K + 20L$ subject to $\sqrt{K} + L = 29.9$, $K \geq 0$, $L \geq 0$.

3.4 A Second-Order Criterion for a Constrained Optimum

As already noted in the introduction to this chapter, optimizing a function $f(x, y)$ subject to the constraint $g(x, y) = b$ essentially amounts to optimizing the restriction of $f(x, y)$ to a constrained domain. In other words, we seek the extrema of a scalar function whose domain has a more complex structure. In the previous section, we showed that these optima can typically be identified as stationary points of the associated Lagrange function. Since the problem reduces to optimizing a function over a restricted domain, it is natural to expect that a suitable second-order criterion would allow us to classify the stationary points obtained via Lagrange's method. One might therefore hope that a direct examination of the second-order derivatives of the Lagrange function at the corresponding stationary points would yield conclusions about the candidate solutions of the constrained optimization problem (3.1). However, the following example demonstrates that this is not true in general.

EXAMPLE 3.22 Consider the maximization of $f(x, y) = xy$ on the constraint set $x + y = 2$. Substituting $y = 2 - x$ into $f(x, y)$ shows that the problem is equivalent to maximizing the scalar function

$$F(x) := x(2 - x).$$

It is easily verified that $F(x)$ attains a global maximum at $x = 1$. Therefore, the constrained optimization problem has a global maximum at $(x, y) = (1, 1)$.

Next, consider the stationary points of the associated Lagrange function

$$L(x, y, \lambda) = xy - \lambda(x + y - 2).$$

Its derivative is

$$L'(x, y, \lambda) = [y - \lambda \quad x - \lambda \quad x + y - 2].$$

It is easily verified that the only solution to $L'(x, y, \lambda) = [0 \ 0 \ 0]$ is $(x, y, \lambda) = (1, 1, 1)$.

However, $(x, y) = (1, 1)$ does not maximize the Lagrange function $L(x, y, 1) = xy - x - y + 2$. A direct inspection of the second-order derivatives of $L(x, y, 1)$ show that $L(x, y, 1)$ has a saddle-point at $(x, y) = (1, 1)$.

Upon closer reflection, it becomes clear that examining the second-order derivatives of the Lagrange function is, in general, not very meaningful. From a geometric perspective, we are optimizing a function restricted to an intersection set, which can vary only along directions permitted by its constrained domain. Consequently, to determine whether a stationary point corresponds to an optimum, one must consider only those function values that the restricted (intersection) function can attain in a neighborhood of that point. Therefore, in order to derive a second-order criterion for the constrained optimization problem, we return to the original approach. Specifically, by applying the Implicit Function Theorem, we once again reformulate the constrained optimization problem (3.1) as the scalar optimization problem:

$$\text{optimize } F(x) := f(x, h(x)), \text{ where } y = h(x) \text{ solves } g(x, y) = b.$$

From this, we conclude that the constrained optimization problem attains an optimum at (c, d) if and only if the function $F(x)$ attains an optimum at $x = c$. Consequently, by examining the second-order derivative of F at $x = c$, we can obtain further insight into the nature of the optimum achieved at the stationary point c . Accordingly, we now compute this second-order derivative of the intersection function. To this end, we first recall that the derivative of F is given by:

$$F'(x) = D_1 f(x, h(x)) + D_2 f(x, h(x))h'(x).$$

Therefore, the second-order derivative of F equals (dropping the arguments):

$$F''(x) = D_{11}f + D_{12}fh' + D_{21}fh' + D_{22}f(h')^2 + D_2fh''.$$

Since $G(x) := g(x, h(x)) = b$ we have that, for all x satisfying this constraint,

$$G'(x) = D_1g(x, h(x)) + D_2g(x, h(x))h'(x) = 0.$$

Consequently (dropping the arguments),

$$G''(x) = D_{11}g + D_{12}g * h' + D_{21}g * h' + D_{22}g * (h')^2 + D_2g * h'' = 0.$$

From this, after some elementary matrix manipulations, we deduce that

$$h''(x) = -(D_{22}g)^{-1} * [1 \ h'] \begin{bmatrix} D_{11}g & D_{12}g \\ D_{21}g & D_{22}g \end{bmatrix} \begin{bmatrix} 1 \\ h' \end{bmatrix}.$$

Substitution of h'' into F'' yields:

$$F''(x) = [1 \ h'] \begin{bmatrix} D_{11}f & D_{12}f \\ D_{21}f & D_{22}f \end{bmatrix} \begin{bmatrix} 1 \\ h' \end{bmatrix} - \frac{D_2f}{D_{22}g} * [1 \ h'] \begin{bmatrix} D_{11}g & D_{12}g \\ D_{21}g & D_{22}g \end{bmatrix} \begin{bmatrix} 1 \\ h' \end{bmatrix}.$$

Recall that, when we introduced the Lagrange function, $\lambda = \frac{D_2f}{D_{22}g}$. Consequently, introducing $B := [1 \ h']^T$ (note: $h' = -\frac{D_{12}g}{D_{22}g}$, see the Implicit Function Theorem), F'' can be rewritten as:

$$F''(x) = B^T(Hf - \lambda Hg)B. \quad (3.9)$$

Since $F(x)$ is a scalar function, we can apply the second-order criterion from Chapter 1 to determine when a stationary point corresponds to an optimum. In particular, Theorem 1.40 implies that the constrained optimization problem has a local maximum at (c, d) if $F''(x) \leq 0$ in a neighborhood of $x = c$. Unfortunately, because we don't have an explicit description of $h(x)$, this condition is difficult to verify in practice. However, Theorem 2.54 provides a more tractable criterion: if $F''(c) < 0$, the constrained optimization problem has a local maximum at (c, d) , whereas if $F''(c) > 0$, it has a local minimum at (c, d) . These conditions are easier to check, since we know that $h(c) = d$.

This second-order condition on F can be reformulated in terms of the Hessian of the Lagrange function $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$:

$$HL = \begin{bmatrix} D_{11}f - \lambda D_{11}g & D_{12}f - \lambda D_{12}g & -D_{1g} \\ D_{21}f - \lambda D_{21}g & D_{22}f - \lambda D_{22}g & -D_{2g} \\ -D_{1g} & -D_{2g} & 0 \end{bmatrix}.$$

The condition $F'''(c) < 0$ ($F'''(c) > 0$) is equivalent to the condition $\det HL > 0$ ($\det HL < 0$) at the stationary point - note the change in sign.

In the reasoning above, we assumed that, in a neighborhood of the optimum (c, d) , the variable y could be expressed as a function of x along the constraint set. If this is not possible, but x can instead be solved from the constraint set as a function of y , we can proceed analogously. This yields the next second-order condition for a constraint optimum.

THEOREM 3.23 (second-order criterion of a constrained optimum, 2 variables and one constraint)

Consider the constrained optimization problem (3.8) with $f, g \in C^2$. Assume $g'(c, d) \neq [0 \ 0]$ and (c, d, λ_0) is a stationary point of the corresponding Lagrange function. Let

$$HL := \begin{bmatrix} Hf - \lambda^* Hg & -(Dg)^T \\ -Dg & 0 \end{bmatrix}.$$

Then, $f(c, d)$ will be

- a local maximum if $\det HL(c, d, \lambda_0) > 0$;
- a local minimum if $\det HL(c, d, \lambda_0) < 0$.

REMARK 3.24 In the literature, this second-order criterion is also referred to as the bordered Hessian test. We emphasize that it applies only to optimization problems with two variables and a single constraint. The generalization of this second-order criterion to problems with more variables and multiple constraints is discussed in the next section.

EXAMPLE 3.25 Consider the constrained optimization problem

$$\text{optimize } f(x, y) = x + y^2 \text{ subject to } g(x, y) := x^2 + y^2 = 1.$$

It is easily verified that the constraint set is a bounded and closed set, i.e., it is compact. Furthermore, f is continuous. Therefore, by Weierstrass' theorem, we know that the problem has a global maximum and a global minimum.

To illustrate the application of Algorithm 3.13 for solving constraint optimization problems, we follow its steps below to determine the optima. Note, however, that this example can be solved very easily too using the substitution method. Substitution of $y^2 = 1 - x^2$ into the objective function shows that the constrained optimization problem is equivalent with determining the optima of $F(x) = x + 1 - x^2$ on the interval $[-1, 1]$. A simple analysis shows that F has a global maximum at $x = \frac{1}{2}$, a global minimum at $x = -1$ and a local minimum at $x = 1$.

Applying Algorithm 3.13 step by step, we find:

Step 1. $f' = [1 \ 2y]$ and $g' = [2x \ 2y]$. Clearly, all partial derivatives are continuous. So, $f, g \in C^1$.

Step 2. $g'(x, y) = [0 \ 0]$ only in case $(x, y) = (0, 0)$. However, $g(0, 0) = 0 \neq 1$. So, $(0, 0)$ is not in the feasible region. We, therefore, don't find any candidate solutions in this step.

Step 3. The Lagrange function is $L(x, y, \lambda) = x + y^2 - \lambda(x^2 + y^2 - 1)$.

Step 4. Differentiation of the Lagrange function shows that the candidate solutions must satisfy

$$L' = [1 - 2\lambda x \ 2y - 2\lambda y \ x^2 + y^2 - 1] = [0 \ 0 \ 0].$$

From the second equation we have that $2y(1 - \lambda) = 0$. So, either $y = 0$ or $\lambda = 1$. If $y = 0$, from $g(x, y) = 1$, $x = 1$ or $x = -1$. This yields the stationary points $(1, 0, \frac{1}{2})$ and $(-1, 0, -\frac{1}{2})$, respectively. If $\lambda = 1$, it follows from the first equation that, $\lambda = 1$. From the third equation, we obtain then the two additional stationary points $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1)$.

Since $f(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1) = f(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1) = \frac{5}{4}$, $f(1, 0) = 1$ and $f(-1, 0) = -1$, we conclude from Weierstrass theorem that the constrained optimization problem has a global maximum at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, respectively, and a global minimum at $(-1, 0)$.

To assess what kind of optimum occurs at $(1, 0)$, we use the second-order criterion we formulated in Theorem 3.23. To that end, first note that, $f, g \in C^2$ and $g'(1, 0) \neq [0 \ 0]$. Simple calculations show that

$$HL := \begin{bmatrix} -2\lambda & 0 & -2x \\ 0 & -2\lambda & -2y \\ -2x & -2y & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}.$$

The determinant of $HL(1, 0, \frac{1}{2}) = -4 < 0$. So, by Theorem 3.23 the constrained optimization problem has a local minimum at $(1, 0)$. Note this is not a global minimum as $f(1, 0) > f(-1, 0)$.

EXERCISE 3.12 Solve next constrained optimization problems. Optimize

- $f(x, y) = x^2 - y$ subject to $g(x, y) = x^2 - y^2 = 1$.
- $f(x, y) = 2x^3 + 6y^3$ subject to $g(x, y) = 3x^2 + 9y^2 = 12$.

EXERCISE 3.13 Solve next constrained optimization problems.

- Optimize $f(x, y) = x^2 + 4y^2 - x^2y$ subject to $g(x, y) = x^2 + y^2 = 3$.
- Optimize $f(x, y, z) = x^2 + y^2 + z^2 - 5x$ subject to $g(x, y) = 3x + 2y - 10z = 0$.
- Optimize $f(x, y) = xy + y$ subject to $g(x, y) = x^2 + y^2 = 1$.
- Minimize $f(x, y) = x + 2y$ subject to $g(x, y) = x^2 + y^2 = \frac{5}{4}$, $x \geq 0$, $y \geq 0$.

3.5 The Method of Lagrange: the general case

In this section we consider the general constrained optimization problem (3.1). That is, if the objective function is a function of more than two variables and there are one or more "budget" constraints that describe the feasible region. So, the problem is to find the optimal values of the objective function $f(x_1, \dots, x_n)$ on the *feasible region* $D \subset \mathbb{R}^n$, consisting of the intersection of the domain of f and all points which satisfy the system of equations

$$\begin{cases} g_1(x_1, \dots, x_n) &= b_1, \\ &\vdots \\ g_m(x_1, \dots, x_n) &= b_m. \end{cases}$$

To prevent that the feasible region consists of only one point or is even empty, in the analysis below we assume that the number of constraints is less than the number of variables, i.e., $m < n$.

The solution procedure follows the lines of the case $n = 2, m = 1$, we analyzed in the previous sections. That is, from the m equality constraints we solve, either explicitly or implicitly, m variables as a function of the remaining $n - m$ variables. Next, we substitute these functions into our objective function. Our new objective function is then a function of $n - m$ variables, and we have to find all optima of this function (without any further constraints on its domain). This latter optimization can be carried out using the methods we developed in Chapter 2. We will not present a detailed analysis here, but just present the main results and illustrate them in a number of examples.

From the preceding discussion, it should be clear that the first result required to solve the optimization problem is an extension of the Implicit Function Theorem. That is, a result which tells us under which conditions we can solve from m equations, m variables as a function of the remaining $n - m$ variables. Without loss of generality, we assume that the m variables to be solved as functions of the remaining variables are $x_1, \dots, x_m$, which we collectively denote by x . The remaining variables, $x_{m+1}, \dots, x_n$ are denoted by y . The problem can be rephrased then as: find conditions under which from the set of m equations

$$g(x, y) = b, \text{ where } x \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}, b \in \mathbb{R}^m,$$

x can be solved as a function of y . Conditions under which this can be accomplished are stated in the next theorem. In this theorem, we use the next shorthand notation for the derivative of g : $Dg = [D_x g \ D_y g]$, where $D_x g$ denotes the matrix of partial derivatives of g with respect to the first m variables, and $D_y g$ the matrix of partial derivatives with respect to the remaining $n - m$ variables.

THEOREM 3.26 (*Implicit Function Theorem, general case*)

Let $g(x, y)$ be a continuously differentiable function in a ball around (c, d) . Suppose $g(c, d) = b$. Then, if

$$D_x g(c, d) \text{ is invertible,}$$

there exists, in a ball B around $y = d$, a differentiable function h , such that $x = h(y)$ is the solution of $g(x, y) = b$. Moreover, $h'(y) = -[D_x g]^{-1} * D_y g$.

EXAMPLE 3.27 Consider the equation

$$g(x, \alpha, \beta) = x^3 - \alpha^3 + x\alpha\beta \ln(\beta) = 0 \text{ at } (x, \alpha, \beta) = (1, 1, 1).$$

Then,

$$g'(x, \alpha, \beta) = [3x^2 + \alpha\beta \ln(\beta) \quad -3\alpha^2 + x\beta \ln(\beta) \quad \alpha x(\ln(\beta) + 1)].$$

All partial derivatives are continuous in a ball around $(1, 1, 1)$. So, in a ball around $(1, 1, 1)$, $g \in C^1$. Furthermore, $D_1 g(1, 1, 1) = 3 \neq 0$ (and thus invertible). Consequently, by Theorem 3.26, there exists a differentiable function $h(\alpha, \beta)$ such that $x = h(\alpha, \beta)$.

Moreover in a ball around $(1, 1)$, the derivative of h equals:

$$h'(\alpha, \beta) = -\frac{1}{3x^2 + \alpha\beta \ln(\beta)} [-3\alpha^2 + x\beta \ln(\beta) \quad \alpha x(\ln(\beta) + 1)].$$

In particular, at $(1, 1)$ we have $h'(1, 1) = [-1 \quad \frac{1}{3}]$.

EXAMPLE 3.28 Consider the set of equations

$$g_1(x_1, x_2, y) = x_1^2 - x_2^2 - \ln(x_1 y) = 0 \text{ and } g_2(x_1, x_2, y) = \sqrt{x_1 + y} - \sqrt{2} \frac{y}{x_2} = 0,$$

at $(x_1, x_2, y) = (1, 1, 1)$. Then,

$$g'(x_1, x_2, y) = \begin{bmatrix} 2x_1 - \frac{1}{x_1} & -2x_2 & -\frac{1}{y} \\ \frac{1}{2} \frac{1}{\sqrt{x_1 + y}} & \sqrt{2} \frac{y}{x_2^2} & \frac{1}{2} \frac{1}{\sqrt{x_1 + y}} - \sqrt{2} \frac{1}{x_2} \end{bmatrix}.$$

All partial derivatives are continuous in a ball around $(1, 1, 1)$. So, in a ball around $(1, 1, 1)$, $g \in C^1$. Furthermore,

$$D_x g(1, 1, 1) = \begin{bmatrix} 1 & -2 \\ \frac{1}{2\sqrt{2}} & \sqrt{2} \end{bmatrix}.$$

Note that, $\det D_x g(1, 1, 1) \neq 0$. So, $D_x g(1, 1, 1)$ is invertible.

Consequently, by Theorem 3.26, in a neighborhood of $y = 1$ variables x_1 and x_2 can be solved as differentiable functions $h_1(y)$ and $h_2(y)$ of the variable y , i.e., $x_1 = h_1(y)$ and $x_2 = h_2(y)$ in a neighborhood of $y = 1$.

The derivative of h equals:

$$h'(y) = \begin{bmatrix} h'_1(y) \\ h'_2(y) \end{bmatrix} = -[D_x g]^{-1} * D_y g = - \begin{bmatrix} 2x_1 - \frac{1}{x_1} & -2x_2 \\ \frac{1}{2} \frac{1}{\sqrt{x_1+y}} & \sqrt{2} \frac{y}{x_2^2} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{y} \\ \frac{1}{2} \frac{1}{\sqrt{x_1+y}} - \sqrt{2} \frac{1}{x_2} \end{bmatrix}.$$

In particular, at $(1, 1)$ we have

$$\begin{aligned} h'(1) &= \begin{bmatrix} h'_1(1) \\ h'_2(1) \end{bmatrix} = - \begin{bmatrix} 1 & -2 \\ \frac{1}{2\sqrt{2}} & \sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ \frac{1}{2\sqrt{2}} - \sqrt{2} \end{bmatrix} \\ &= -\frac{1}{\sqrt{2} + \frac{1}{\sqrt{2}}} \begin{bmatrix} \sqrt{2} & 2 \\ -\frac{1}{2\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1}{2\sqrt{2}} - \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}. \end{aligned}$$

EXAMPLE 3.29 Consider the set of equations

$$g_1(x_1, x_2, y_1, y_2) = x_1^2 + x_2 - y_1 = 0 \text{ and } g_2(x_1, x_2, y_1, y_2) = x_1 - x_2^2 - y_2 = 0,$$

at $(x_1, x_2, y_1, y_2) = (1, 1, 2, 0)$. Then,

$$g'(x_1, x_2, y_1, y_2) = \begin{bmatrix} 2x_1 & 1 & -1 & 0 \\ 1 & -2x_2 & 0 & -1 \end{bmatrix}.$$

All partial derivatives are continuous in a ball around $(1, 1, 2, 0)$. So, in a ball around $(1, 1, 2, 0)$, $g \in C^1$. Furthermore,

$$D_x g(1, 1, 2, 0) = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Note that, $\det D_x g(1, 1, 2, 0) \neq 0$. So, $D_x g(1, 1, 2, 0)$ is invertible.

Consequently, by Theorem 3.26, there exists a ball around $(y_1, y_2) = (2, 0)$ where the set of points satisfying $g_1 = g_2 = 0$ can be described by the set of points (x_1, x_2, y_1, y_2) where $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$, and both $h_1(y_1, y_2)$ and $h_2(y_1, y_2)$ are differentiable functions.

The derivative of h equals:

$$h'(y_1, y_2) = \begin{bmatrix} h'_1(y_1, y_2) \\ h'_2(y_1, y_2) \end{bmatrix} = -[D_x g]^{-1} * D_y g = - \begin{bmatrix} 2x_1 & 1 \\ 1 & -2x_2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In particular, at $(2, 0)$ we have

$$h'(2, 0) = \begin{bmatrix} h'_1(2, 0) \\ h'_2(2, 0) \end{bmatrix} = - \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{-2}{5} \end{bmatrix}.$$

EXAMPLE 3.30 Consider a producer who wants to maximize his profit. His output is described by a production function of the type "decreasing returns to scale", i.e.,

$$q = Q(L), \text{ where } Q'(L) > 0 \text{ and } Q''(L) < 0.$$

Here L denotes the amount of labor used in the production process. Let $Q''(L)$ be continuous. Assuming price, p , for the product, price of labor, ω , price of capital, r , and the amount of used capital, $K = K_0$, are fixed, his profit is given by

$$\pi(L) = pQ(L) - \omega L - rK_0.$$

The optimal quantity of labor, L^* , is determined as the solution of the equation

$$\pi'(L) = pQ'(L) - \omega = 0.$$

To examine whether the optimal quantity of labor depends smoothly on the prices of the product and labor, we regard this equation as a function of the three variables L , p and ω , that is,

$$g(L, p, \omega) = pQ'(L) - \omega = 0.$$

Then,

$$g'(L, p, \omega) = [pQ''(L) \quad Q'(L) \quad -1].$$

Note that all partial derivatives are continuous. Therefore, $g \in C^1$. Since, by assumption, $Q''(L) < 0$, it follows in particular that $Q''(L_0) \neq 0$. Consequently, by Theorem 3.26, in a neighborhood of the optimal labor choice, labor can be expressed as a differentiable function of the price p and the wage ω . Moreover, by this Theorem, the derivative of this function in a neighborhood of L^* is given by

$$DL(p, \omega) = -\frac{1}{pQ''(L)}[Q'(L) \quad -1].$$

EXERCISE 3.14

a. Consider the equation

$$g(x, s, t, u) = e^{x+s+t+u} - x^2 - s^2 - t^2 - u^2 = 1 \text{ at } (x, s, t, u) = (0, 0, 0, 0).$$

Show that from all points (x, s, t, u) satisfying $g(x, s, t, u) = 1$, the variable x is a differentiable function of s , t and u in a neighborhood of $(0, 0, 0)$. Determine its derivative. In particular calculate the derivative at $(0, 0, 0)$.

b. Consider the equations

$$g_1(x, y, z) = xy + yz + xz = 11 \text{ and } g_2(x, y, z) = x^2 - xyz = -5.$$

Show from all points satisfying both these equations the variables x and y are a differentiable function of z in a neighborhood of $z = 3$ and $(x, y) = (1, 2)$. Determine both derivatives. In particular calculate both derivatives at $z = 3$.

c. Reconsider item a). Can variable u also be written as a differentiable function $u = U(x, s, t)$ in a neighborhood of $(x, s, t) = (0, 0, 0)$? Motivate your answer.

To solve a constrained optimization problem with a single constraint in two variables using the substitution method, we had to assume that either x can be locally expressed as a function of y , or conversely, that y can be expressed as a function of x . By applying the Implicit Function Theorem, this translates into the condition that either $D_2g \neq 0$ or $D_1g \neq 0$ at the optimum - or, equivalently, $Dg \neq [0 \ 0]$.

In the general case, the same problem arises: we need to assume that at the optimum, m variables can be locally written as a differentiable function of the remaining $n - m$ variables. The Implicit Function Theorem 3.26 ensures this if the first $m \times m$ block of Dg is invertible. More broadly, it is enough to identify any m columns of Dg forming an invertible $m \times m$ submatrix. By appropriately relabeling the variables, the Implicit Function Theorem can again be applied to conclude that the m variables corresponding to the chosen columns can be locally expressed as a differentiable function of the remaining variables. For example, if we have 3 equations in 5 variables and columns 2, 3, and 5 of Dg form an invertible matrix, then the second, third, and fifth variables can locally be written as differentiable functions of the first and fourth variables.

EXAMPLE 3.31 Consider the set of equations

$$g_1(x, y, s, t, u) = x^2 + y^2 + s^2 + t^2 + u^2 = 5 \text{ and } g_2(x, y, s, t, u) = x^2 - y^2 + s^2 - t^2 + u^2 = 1$$

at $(x, y, s, t, u) = (1, 1, 1, 1, 1)$. Then,

$$Dg(x, y, s, t, u) = \begin{bmatrix} 2x & 2y & 2s & 2t & 2u \\ 2x & -2y & 2s & -2t & 2u \end{bmatrix}.$$

So,

$$Dg(x, y, s, t, u) = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 & 2 \end{bmatrix}.$$

Then, the first two columns of this matrix yield the matrix $\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$, which is clearly invertible. Thus, x and y can locally be expressed as differentiable functions of s , t , and u . Similarly, columns 3 and 4 also form an invertible matrix, so s and t can locally be expressed as differentiable functions of x , y and u . Note, for example, that the first and third columns of this matrix form the matrix

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

which is not invertible. Therefore, we cannot conclude from the Implicit Function Theorem that x and s can be expressed locally as differentiable functions of y , t and u .

The observation above leads to the next requirement on the derivative matrix Dg : From this matrix, it must always be possible to select an invertible $m \times m$ submatrix - that is, to find m columns that together form an invertible matrix. This condition is commonly expressed by stating that Dg has *full row rank*. Equivalently, one can say that the m vectors of derivatives of the constraints, $Dg_i^T, i = 1, \dots, m$, are linearly independent. In literature this requirement is called the *nondegenerate* or *linear independence constraint qualification*⁹. If at a point a in the feasible region this condition is satisfied, a will be called a *regular* point. To verify this condition, notice that an $n \times m$ matrix A has not full row rank if and only if for every $m \times m$ submatrix A_S of A it holds that the determinant of A_S is zero. Another way to verify that the matrix A has full row rank is to either check, using Gaussian elimination, whether A has m pivots, or to verify that the determinant of AA^T is nonzero. Essentially, by following the same reasoning as in the case of a single constraint with two variables, we arrive at the following generalization of Lagrange's theorem.

THEOREM 3.32 (*Lagrange's Theorem, general case*)

Consider the constrained optimization problem (3.1). Assume both $f, g \in C^1$. Assume the problem has a solution at $x = a$. Then, if a is a regular point, there exist $\lambda_i \in \mathbb{R}, i = 1, \dots, m$, such, that

$$f'(a) - \sum_{i=1}^m \lambda_i g'_i(a) = [0 \ \dots \ 0]. \quad (3.10)$$

That is, at the optimum the gradient of the objective function is a linear combination of the gradients of the constraint functions.

⁹ In literature also other conditions on the constraint set are known under which above theorem applies. In Section 4.7 we will elaborate on this using a somewhat different perspective to solve the problem.

REMARK 3.33

1. The above result can be trivially extended for the case that the number of variables, n , equals the number of constraints, m . For, if in that case f has a maximum at $x = a$ and $g'(a)$ is full row rank, some elementary matrix manipulations show that equation (3.10) has the solution $\lambda = [g'^T(a)]^{-1}f'^T(a)$. Here $\lambda = [\lambda_1 \cdots \lambda_n]^T$.
2. Theorem 3.32 implies that to find the optimal solution for the problem one has, basically, to solve a set of $n + m$ equations ((3.10) and $g(x) = 0$) for the $n + m$ variables (x, λ) .

From Theorem 3.32 next analogue of Algorithm 3.13 results to solve our constrained optimization problem (3.1).

ALGORITHM 3.34 To find the candidate solutions of our constrained optimization problem (3.1) proceed as follows:

Step 1. Verify whether $f, g \in C^1$.

Step 2. Find all points a , where $g'(a)$ is not full row rank and $g(a) = b$.

Step 3. Write down the Lagrange function

$$L(s, \lambda_1, \dots, \lambda_m) := f(s) - \sum_{i=1}^m \lambda_i (g_i(s) - b_i).$$

Step 4. Let $\lambda := [\lambda_1, \dots, \lambda_m]$. Determine all solutions (a, λ) of $L'(a, \lambda) = [0 \cdots 0]$.

All points, a , found in Step 2 and Step 4, above, are candidate solutions for the constrained optimization problem (3.2).

A similar remark applies here as after Algorithm 3.13. In Algorithm 3.34 above, Step 2 is included because at points in the feasible region where the derivative $g'(a)$ does not have full row rank, the Implicit Function Theorem fails (when $m < n$). Since our analysis relies on the applicability of this theorem, we cannot draw conclusions about points where this condition is violated. In particular it may occur that the objective function attains an optimum at one or more of these points. Moreover, when $m = n$ and $g'(a)$ is not invertible, it is unclear whether equation (3.10) admits a solution, and such points therefore also require a more careful examination.

Finally, it is worth noting that if the functions f and g are not continuously differentiable, the entire procedure breaks down, and alternative methods must be used to solve the optimization problem.

EXAMPLE 3.35 Consider the optimization of $f(x, y, z) = x + y + z$, subject to the constraints

$$g_1(x, y, z) = x - y - z = 0 \text{ and } g_2(x, y, z) = x^2 + y^2 + z^2 = 6.$$

Note that f is continuous and that the constraint set is compact. So, by Weierstrass' theorem, the problem has both a global maximum and a global minimum.

To find the candidate solutions we proceed along the lines of Algorithm 3.34.

Step 1. $Df = [1 \ 1 \ 1]$ and $Dg = \begin{bmatrix} 1 & -1 & -1 \\ 2x & 2y & 2z \end{bmatrix}$. Since all partial derivatives are continuous both f and g are continuous differentiable.

Step 2. Dg is not full row rank if as well

$$\det \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix} = 0, \det \begin{bmatrix} 1 & -1 \\ 2x & 2z \end{bmatrix} = 0 \text{ and } \det \begin{bmatrix} -1 & -1 \\ 2y & 2z \end{bmatrix} = 0.$$

That is, if simultaneously, $y + x = 0$, $z + x = 0$ and $y - z = 0$. This is the case if and only if $y = z$ and $x = -z$. To see whether there are any points in our constraint set that satisfy these conditions, we substitute these conditions into g_1 and g_2 . Substitution of $y = z$ and $x = -z$ into g_1 shows that $-z - z - z = 0$, i.e., $z = 0$. But this implies that the only candidate is $(0, 0, 0)$. However, $g_2(0, 0, 0) = 0 \neq 6$. So, there are no points where the full rank condition is not satisfied.

Step 3. The Lagrange function is

$$L(x, y, z, \lambda_1, \lambda_2) = x + y + z - \lambda_1(x - y - z) - \lambda_2(x^2 + y^2 + z^2 - 6).$$

Step 4. Differentiation of the Lagrange function gives

$$DL = \begin{bmatrix} 1 - \lambda_1 - 2\lambda_2x & 1 + \lambda_1 - 2\lambda_2y & 1 + \lambda_1 - 2\lambda_2z \end{bmatrix}$$

So, the candidate solutions satisfy: $DL = 0$; $g_1 = 0$ and $g_2 = 6$. That is

- i) $1 - \lambda_1 - 2\lambda_2x = 0.$
- ii) $1 + \lambda_1 - 2\lambda_2y = 0.$
- iii) $1 + \lambda_1 - 2\lambda_2z = 0.$
- iv) $x - y - z = 0.$
- v) $x^2 + y^2 + z^2 = 6.$

Subtracting ii) from iii) yields $\lambda_2(y - z) = 0$. So, either $\lambda_2 = 0$ or $y = z$.

First consider the case: $\lambda_2 = 0$.

Substitution of $\lambda_2 = 0$ into i), ii) and iii) shows that there is no solution in this case.

Next consider the case: $z = y$.

From iv) we have $x = 2y$. Substitution of $z = y$ and $x = 2y$ into v) gives: $y^2 = 1$, i.e., either $y = 1$ or $y = -1$. The stationary points we find are therefore $(2, 1, 1)$, with $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = \frac{1}{3}$, and $(-2, -1, -1)$, with $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = -\frac{1}{3}$.

Finally, we apply Weierstrass's theorem to derive conclusions regarding the stationary points. Since $f(2, 1, 1) > f(-2, -1, -1)$, the constrained optimization problem has a global maximum in $(2, 1, 1)$ and a global minimum in $(-2, -1, -1)$.

By applying the Envelope Theorem 3.17 we again obtain - analogous to Corollary 3.19 and under suitable regularity conditions - that the Lagrange multiplier λ_i equals $\frac{df(x)}{db_i}$ at the stationary point a of the Lagrange function. In other words, if the "budget", b_i , in the constraint $g_i(x) = b_i$ is increased by one unit, the value of f at the corresponding new optimum changes approximately by λ_i units. In economic terminology, this means that the shadow price of the i^{th} budget constraint equals λ_i . The formal statement of this result is given in Corollary 3.36 below.

COROLLARY 3.36 (*Shadow price Interpretation Lagrange multiplier*)

Consider the constrained optimization problem (3.1) with $f, g \in C^2$.

Assume that $(a, \lambda_1, \dots, \lambda_m)$ is a stationary point of the corresponding Lagrange function, where a is a regular point.

Then, if the Hessian of this Lagrange function

$$HL := \begin{bmatrix} Hf - \sum_{i=1}^m \lambda_i Hg_i & -Dg^T \\ -Dg & 0 \end{bmatrix}$$

is invertible at $(a, \lambda_1, \dots, \lambda_m)$, it follows that

$$\frac{df(x)}{db_i} = \lambda_i \text{ at } x = a.$$

EXAMPLE 3.37 Reconsider Example 3.35. In this example, matrix

$$Dg = \begin{bmatrix} 1 & -1 & -1 \\ 2x & 2y & 2z \end{bmatrix}; Hf = Hg_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Hg_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore,

$$HL = \begin{bmatrix} Hf - \sum_{i=1}^m \lambda_i Hg_i & -Dg^T \\ -Dg & 0 \end{bmatrix} = \begin{bmatrix} -2\lambda_2 & 0 & 0 & -1 & -2x \\ 0 & -2\lambda_2 & 0 & 1 & -2y \\ 0 & 0 & -2\lambda_2 & 1 & -2z \\ -1 & 1 & 1 & 0 & 0 \\ -2x & -2y & -2z & 0 & 0 \end{bmatrix}.$$

Consequently, at the maximum $(2, 1, 1)$, where $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = \frac{1}{3}$,

$$HL(2, 1, 1, -\frac{1}{3}, \frac{1}{3}) = \begin{bmatrix} -\frac{2}{3} & 0 & 0 & -1 & -4 \\ 0 & -\frac{2}{3} & 0 & 1 & -2 \\ 0 & 0 & -\frac{2}{3} & 1 & -2 \\ -1 & 1 & 1 & 0 & 0 \\ -4 & -2 & -2 & 0 & 0 \end{bmatrix}.$$

Since $\det(HL) = -48 \neq 0$, we conclude that HL is invertible at the maximum location. Consequently, if b_1 changes from 0 to 0.1, the maximum will approximately change from 4 to $4 - 0.1 * \frac{1}{3}$. In case b_2 changes from 6 to 6.2, the maximum will approximately change from 4 to $4 + 0.2 * \frac{1}{3}$.

Note, if the regularity condition is satisfied, a simultaneous change in the "budgets" from b_i to $b_i + db_i$ implies that the optimum value of the objective function, $\bar{f}(a)$, changes approximately to

$$\bar{f}(a) + \sum_{i=1}^m \lambda_i db_i.$$

Thus, for example, in the scenario of Example 3.37, if b_1 increases from 0 to 0.1 and b_2 increases from 6 to 6.2, the maximum value will approximately change from 4 to $4 - 0.1 * \frac{1}{3} + 0.2 * \frac{1}{3}$.

Along the lines of the arguments presented in Section 3.4, and invoking Theorem 2.55 and Corollary 2.56, we arrive at the following sufficient conditions for determining whether stationary points are extremal in the general case.

THEOREM 3.38 (sufficient second-order condition for a constrained optimum)

Consider the constrained optimization problem (3.1) with $f, g \in C^2$ (and $m < n$). Assume $(a, \lambda_1, \dots, \lambda_m)$ is a stationary point of the corresponding Lagrange function, where a is regular. Let $g'(a) =: [D_x g(a) \ D_y g(a)]$, where $D_y g(a)$ is the last $m \times m$ block of matrix $g'(a)$. Assume, without loss of generality, that $D_y g(a)$ is invertible. Let

$$H\tilde{L} := Hf - \sum_{i=1}^m \lambda_i H G_i \text{ and } B := \begin{bmatrix} I_{(n-m) \times (n-m)} \\ -(D_y g(a))^{-1} * D_x g(a) \end{bmatrix}.$$

Then, $f(a)$ will be

- a strict local maximum if $B^T * H\tilde{L} * B$ is negative definite.
- a strict local minimum if $B^T * H\tilde{L} * B$ is positive definite.
- a saddle point if $B^T * H\tilde{L} * B$ is indefinite.

Here $I_{(n-m) \times (n-m)}$ denotes the identity matrix in $\mathbb{R}^{(n-m) \times (n-m)}$.

In the above theorem, we can assume without loss of generality that $D_y g(a)$ is invertible. Indeed, the assumption that $g'(a)$ has full row rank guarantees the existence of m columns of $g'(a)$ forming an

invertible matrix. Hence, if the last m columns of $g'(a)$ are not invertible, this can be remedied by a suitable relabeling of the variables.

One way to avoid this relabeling is to define

$$d(v) = v^T H \tilde{L} v \text{ for all vectors } v \text{ satisfying } g'(a)v = 0.$$

If $d(v) < 0$ (respectively, $d(v) > 0$) for all nonzero v , then $f(a)$ is a local maximum (respectively, a local minimum). If $d(v)$ takes both negative and positive values on this set, then $f(a)$ is a saddle-point. Using this notation and arguing analogously to Corollary 2.56, Theorem 3.38 yields the following necessary conditions for the existence of an extremum at a stationary point of the associated Lagrangian.

COROLLARY 3.39 *(necessary second-order condition for a constrained optimum)*

Consider the constrained optimization problem (3.1) with $f, g \in C^2$ (and $m < n$). Assume $(a, \lambda_1, \dots, \lambda_m)$ is a stationary point of the corresponding Lagrange function, where a is regular. Consider $d(v) = v^T H \tilde{L} v$ for all $v \in N$, where $N = \{v \mid g'(a)v = 0\}$ ¹⁰. Then, if $f(a)$ is a

- local maximum, $d(v) \leq 0$ for all $v \in N$;
- local minimum, $d(v) \geq 0$ for all $v \in N$.

EXAMPLE 3.40 Let us reconsider Example 3.35. Using Weierstrass' theorem, we showed in this example that $(2, 1, 1)$ is the global maximum of the constrained optimization problem. We now illustrate how Theorem 3.38 leads to the same, though local, conclusion.

To this end, first note that, using the notation of Theorem 3.38,

$$D_y g = \begin{bmatrix} -1 & -1 \\ 2y & 2z \end{bmatrix} \text{ and } D_x g = \begin{bmatrix} 1 \\ 2x \end{bmatrix}.$$

Therefore,

$$D_y g(2, 1, 1) = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}.$$

Unfortunately, we see that $D_y g(2, 1, 1)$ is not invertible. Therefore, instead of the last two columns of g' , we consider, for example, the first and second columns of g' . Clearly, this matrix is invertible. That is, at the optimum we can rewrite the variables x and y as a function of z . Therefore, with a redefinition of the variables as: $x_1 := z$, $x_2 := x$ and $x_3 := y$, we rewrite the problem as the

¹⁰ In linear algebra this set N is called the nulspace of matrix $g'(a)$. An often used technique to find this nulspace is to use Gaussian elimination (see, e.g., [13]).

optimization of

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \text{ subject to} \\ g_1(x_1, x_2, x_3) &= -x_1 + x_2 - x_3 = 0 \text{ and } g_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = 6. \end{aligned}$$

Obviously, the stationary point of the corresponding Lagrange function is: $(1, 2, 1)$ with $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = \frac{1}{3}$. Furthermore, at this stationary point,

$$Dg = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 4 & 2 \end{bmatrix}; \quad Hf = Hg_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Hg_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore,

$$H\tilde{L}(1, 2, 1, -\frac{1}{3}, \frac{1}{3}) = \begin{bmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}; \quad D_y g = \begin{bmatrix} -1 & -1 \\ 4 & 2 \end{bmatrix} \text{ and } D_x g = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So, with,

$$B := \begin{bmatrix} 1 \\ \begin{bmatrix} -1 & -1 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \begin{bmatrix} 2 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix},$$

we obtain

$$B^T * H\tilde{L} * B = [1 \ 2 \ -3] \begin{bmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = -\frac{28}{3}.$$

Clearly, $B^T * H\tilde{L} * B$ is negative definite. Therefore, by Theorem 3.38, the stationary point is a local maximum.

To reach this conclusion while avoiding this relabeling of variables, we first determine the nulspace of the matrix $g'(2, 1, 1)$, namely, the nulspace of

$$\begin{bmatrix} 1 & -1 & -1 \\ 4 & 2 & 2 \end{bmatrix}.$$

Some simple calculations show that $g'(2, 1, 1)v = 0$ if and only if $v = \alpha[0 \ 1 \ -1]^T$, with $\alpha \in \mathbb{R}$. Therefore, defining

$$d(v) := v^T H\tilde{L}(2, 1, 1, \frac{1}{3}, \frac{1}{3})v,$$

we have $d(v) = -\frac{8}{3}\alpha^2 < 0$, for all $\alpha \neq 0$. Hence, $d(v)$ is indeed negative on the nulspace of $g'(2, 1, 1)$ (excluding the zero vector).

EXERCISE 3.15 Solve the next constrained optimization problems.

- Optimize $f(x, y, z) = x - y + z$ subject to the constraints $g_1(x, y, z) = x^2 + 2y^2 + z^2 = 40$ and $g_2(x, y, z) = x + y + z = 0$. Moreover, calculate by how much the global maximum will, approximately, change if, $b_1 = 40$, is decreased to, $b_1 = 39$.
- Optimize $f(x, y, z) = x^2 - y + z$ subject to the constraints $g_1(x, y, z) = x^2 + y^2 = 5$ and $g_2(x, y, z) = x^2 + z = 0$. Moreover, calculate by how much the global minimum will, approximately, change if, $b_1 = 5$, is decreased to, $b_1 = 4.9$.
- Optimize $f(x, y, z) = x^2 + y - z$ subject to the constraints $g_1(x, y, z) = x^2 + y - z = 0$ and $g_2(x, y, z) = \frac{2}{3}x^3 + \frac{1}{2}y^2 + z = 0$.

Mixed Exercises

EXERCISE 3.16 Determine all optima (location, value, kind (local/global)) of the next constrained optimization problems.

- Optimize $f(x, y) = xy^2$, subject to the constraint $g(x, y) = x + y = 1$.
- Optimize $f(x, y) = (1 - y)e^x$, subject to the constraint $g(x, y) = x^3 + y = 1$.
- Optimize $f(x, y, z) = \frac{1}{2}x^2y + z$ subject to the constraints $g_1(x, y, z) = \frac{1}{2}x^2 + 1 - y = 0$ and $g_2(x, y, z) = \frac{4}{3}x - \frac{4}{3}xy - z = 0$.
- Optimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraints $g_1(x, y, z) = x + y + z = 0$ and $g_2(x, y, z) = x - y + z = 2$.
- Optimize $f(x, y) = xy^2$, subject to the constraint $g(x, y) = x + y^2 = 1$.
- Optimize $f(x, y, z) = e^{xyz}$, subject to the constraint $g(x, y, z) = xy - z = 0$.
- Optimize $f(s, t, u, v) = s(u + v) + t(u - v)$, subject to the constraints $g_1(s, t, u, v) = s - u - v = 0$ and $g_2(s, t, u, v) = s + t - 2u = 0$.
- Optimize $f(s, t, u, v) = (s - t)(u - v)$, subject to the constraint $g_1(s, t, u, v) = s - t - u + v = 0$.

EXERCISE 3.17 Determine all optima (location, value, kind (local/global)) of the next constrained optimization problems.

- Optimize $f(x, y) = xy^2$, subject to the constraint $g(x, y) = x^2 + y^2 = 12$.
- Optimize $f(x, y) = e^x y^2$, subject to the constraint $g(x, y) = x + y^2 = 1$.
- Optimize $f(x, y) = x^2 y$, subject to the constraint $g(x, y) = x^2 - y^2 = 1$.
- Optimize $f(x, y, z) = x^2 z$ subject to the constraints $g_1(x, y, z) = x^2 + y^2 + z = 1$ and $g_2(x, y, z) = x^2 + y = 1$.
- Optimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraints $g_1(x, y, z) = x^2 + y^2 = 4$ and $g_2(x, y, z) = y^2 + z^2 = 2$.

- f. Optimize $f(x, y, z) = x^2 + \frac{4}{3}y^3 + 16y - 8z^2$, subject to the constraints $g_1(x, y, z) = x^2 + y^4 = 16$ and $g_2(x, y, z) = (y - 2)^2 + z^2 = 9$.
- g. Optimize $f(x, y) = e^{xy}z - 3(1 - x - y)$, subject to the constraint $g_1(x, y, z) = e^{xy} - z = -1$ and $g_2(x, y, z) = x + y + z = 1$.
- h. Optimize $f(x, y, z) = e^{xyz}$, subject to the constraint $g_1(x, y, z) = xy - z = 0$ and $g_2(x, y, z) = x^2 + z^2 = 1$.

EXERCISE 3.18 Use Lagrange's method to determine all points that qualify for an optimum of the next constrained optimization problems. Use, next, second-order criteria; Weierstrass' theorem; or convexity of the Lagrange function (see Exercise 3.24, below) to determine the optima (location, value, kind (local/global)) of the problems.

- a. Optimize $f(x, y) = xy$, subject to the constraint $g(x, y) = x^2 + y^2 = 2$.
- b. Optimize $f(x, y) = x^2y$, subject to the constraint $g(x, y) = x^4 + y^4 = 64$.
- c. Optimize $f(x, y) = 4x + 2y$, subject to the constraint $g(x, y) = x^{\frac{2}{3}}y^{\frac{1}{3}} = 10$, $x > 0$, $y > 0$.
- d. Optimize $f(x, y) = x^2 + y^2$, subject to the constraint $g(x, y) = x^2 + xy + y^2 = 3$.
- e. Optimize $f(x, y) = x^2 + (y - 1)^2$, subject to the constraint $g(x, y) = 4x^2 + y^2 = 16$.
- f. Optimize $f(x, y) = x^2 + \ln(2 + y^2)$, subject to the constraints $g(x, y) = 2x + y^2 = 2$.
- g. Optimize $f(x, y) = e^{xy}$, subject to the constraint $g(x, y) = x^2 + y^2 = 1$.
- h. Optimize $f(x, y) = xye^{\frac{1}{2}x^2+y^2}$, subject to the constraint $g(x, y) = x^2 + y^2 = 4$.

EXERCISE 3.19 Use Lagrange's method to determine all points that qualify for an optimum of the next constrained optimization problems. Use, next, second-order criteria; Weierstrass' theorem; or convexity of the Lagrange function (see Exercise 3.24, below) to determine the optima (location, value, kind (local/global)) of the problems.

- a. Optimize $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$, subject to the constraint $g(x, y) = x^2 + y^2 + z^2 = 1$.
- b. Optimize $f(x, y, z) = x + y + z$, subject to the constraints $g_1(x, y, z) = x^2 + y^2 + z^2 = 12$ and $g_2(x, y, z) = x - y = 0$.
- c. Optimize $f(x, y, u, v) = \ln(1 + xy)$, subject to the constraint $g_1(x, y, u, v) = x^2 + y^2 = 2$, $g_2(x, y, u, v) = xu^2 = 1$ and $g_3(x, y, u, v) = yv^2 = 1$.
- d. Optimize $f(x, y) = x^2 + y^2$, subject to the constraint $g(x, y) = x^2 + xy + y^2 = 3$.
- e. Optimize $f(x, y) = x^2 + (y - 1)^2$, subject to the constraint $g(x, y) = 4x^2 + y^2 = 16$.
- f. Optimize $f(x, y) = x^2 + \ln(2 + y^2)$, subject to the constraints $g(x, y) = 2x + y^2 = 2$.
- g. Optimize $f(x, y) = e^{xy}$, subject to the constraint $g(x, y) = x^2 + y^2 = 1$.
- h. Optimize $f(u, s, t, v) = u + s + t + v$, subject to the constraint $g(u, s, t, v) = u^2 + s^2 + t^2 + v^2 = 4$.

Advanced, Deepening and Broadening Exercises

EXERCISE 3.20 (solving a constrained optimization problem under linear equality constraints)

In this exercise we indicate two ways how one may pursue to solve a constrained optimization problem subject to linear equality constraints.

Consider the optimization of $f(x)$ subject to $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ is full row rank.

- Show that by a simple relabeling the columns of matrix A can be "reshuffled" as $[A_1 \ A_2]$, where matrix A_1 is invertible. Deduct from this that the constrained optimization problem can be solved by finding the optima of the unconstrained function $f(x_2)$, where $[x_1^T \ x_2^T]^T$ is the corresponding "reshuffled" vector x with $x_2 \in \mathbb{R}^{n-m}$ and $x_1 = A_1^{-1}(b - A_2x_2)$.
- Let $A^+ := A^T(AA^T)^{-1}$. Show that the solution set of $\{x \mid Ax = b\}$ equals $\{A^+b + (I - A^+A)y \mid y \in \mathbb{R}^n\}$. Deduct from this that the constrained optimization problem can be solved by finding the optima of the unconstrained function $f(A^+b + (I - A^+A)y)$, $y \in \mathbb{R}^n$.
- Use both methods from item a. and b. to find the optima of the constrained optimization problem: $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ subject to $x_2 + x_3 = 2$ and $x_3 + x_4 = 1$.

EXERCISE 3.21 (Step 2 optimum)

Consider the problem to find the minimum of $f(x, y) = (x+2)^2 + y^2$ subject to the constraint $g(x, y) = x(x+1)^2 - y^2 = 0$.

- Find all candidate extremal points using Lagrange's method.
- Show that $(-1, 0)$ yields the global minimum of the constrained optimization problem.

EXERCISE 3.22 (infinite number of optimum locations)

Consider the problem to find the minimum of $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ subject to the constraints $g_1(x, y, z) = x^2 + y^2 + z^2 = 3$ and $g_2(x, y, z) = x + y + z = 0$.

- Find all extremal points by a direct use of Lagrange's method.
- Use the constraint g_2 to reduce the problem to a constrained optimization problem in two variables. Solve this reduced problem using Lagrange's method. Compare your answer with item a.

EXERCISE 3.23 (Reformulation of strict inequality constraint problem as an equality constraint problem)

Consider the problem to find optima of $f(x, y) = \ln(1+xy)$, subject to the equality constraint $g(x, y) = x^2 + y^2 = 2$ and inequality constraints $x > 0$ and $y > 0$.

- Show that $x > 0$ if and only if there exists u such that $xu^2 = 1$.
- Show that the optimization problem is equivalent to finding the extrema of $f(x, y, u, v) = \ln(1+xy)$, subject to the equality constraints $g_1(x, y, u, v) = x^2 + y^2 = 2$, $g_2(x, y, u, v) = xu^2 = 1$ and $g_3(x, y, u, v) = yv^2 = 1$.
- Solve the optimization problem.

EXERCISE 3.24 (*Stationary point of a convex Lagrange function implies a global constraint minimum*)

Below we show next property. Let f and g be differentiable on an open convex set S . For some fixed λ , the Lagrange function $L(x) = f(x) - \lambda g(x)$ has a stationary point at $x = a$; $g(a) = b$; and $L(x)$ is convex on S . Then $f(x)$, subject to the constraint $g(x) = b$, attains a global minimum at $x = a$.

- Show that $L(a) \leq L(x)$.
- Use item a to show that $f(a) \leq f(x)$ for all x that satisfy $g(x) = b$.
- Formulate a similar sufficient condition under which $x = a$ yields a global maximum of the constrained optimization problem.

EXERCISE 3.25 (*Optimization (homogeneous) function under linear equality constraints*)

Assume $f(x)$ is twice differentiable and the optimization of f subject to the set of linear equality constraints $A^T x = p$ has a solution.

- Show that this solution $x^*(A, p)$ is a homogeneous function of degree zero.
- Assume f is a homogeneous function of some degree $k \geq 0$. Show that the solution of the problem to optimize f subject to the set of linear equality constraints $A^T x = \alpha p$, $\alpha \geq 1$, has a solution x^N that satisfies $A^T x^N = \alpha p$ and $x^N = \alpha x^*$.

EXERCISE 3.26 (*Optimization linear function under linear equality constraints*)

In this exercise we consider the optimization of a linear function $f(x) = a^T x$ subject to linear equality constraints $g(x) := Bx = b$, where $a, x \in \mathbb{R}^n$ and $B \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Without loss of generality assume both a and B have full row rank (which implies in particular that the number of constraints

at most equals n). Let $D := \begin{bmatrix} a^T \\ B \end{bmatrix}$.

- Consider the case $n = 2$ and $m = 1$. Show the problem has an infinite number of solutions iff rank $D = 1$. Show that in that case f has a constant value on the constrained set. Show that, otherwise, the problem has an infimum of $-\infty$ and a supremum of ∞ .
- Consider the case $n = m = 2$. Show that the problem has both a global maximum and minimum that is realized at the same unique point.
- Show that for arbitrary $m \leq n$ the problem has a global minimum iff either $m = n$ or the rank of matrix D equals m . In both situations also the problem has a global maximum, that coincides with the minimum. Show that in the firstmentioned case the location where the optimum is attained is unique and in the lastmentioned case f attains the same value at all points in the constraint set. Show next that if none of both cases applies the problem has an infimum of $-\infty$ and a supremum of ∞ .
- Show the problem has a solution iff rank $D = \text{rank } B$.

EXERCISE 3.27 (*Least-squares optimization under equality constraints*)

The Least-squares approximation problem is the problem to find a vector $x \in \mathbb{R}^m$ such that the distance between a given vector $b \in \mathbb{R}^n$ and Ax is as small as possible. Here A is a given matrix in $\mathbb{R}^{n \times m}$, with $n > m$ which has full column rank. In this exercise you are asked to find the solution to this problem under the additional constraint that vector x should satisfy the set of linear constraints $Ex = d$, where $E \in \mathbb{R}^{k \times m}$ and $d \in \mathbb{R}^k$.

- a. Use Lagrange's theorem to show that necessary conditions for solvability of the problem:

$$\min_x (Ax - b)^T Q (Ax - b) \text{ subject to } Ex = d, \quad (3.11)$$

where $Q > 0$ is some weighting matrix, are

$$\begin{bmatrix} A^T Q A & -E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T Q b \\ d \end{bmatrix}. \quad (3.12)$$

- b. Use Exercise 3.24 to show that the advertized conditions in item a. are sufficient too for the existence of a solution.
- c. Recall that a square matrix B is invertible if the only solution to $By = 0$ is $y = 0$. Show that matrix $A^T Q A$ is invertible.
- d. Show that the least-squares problem has a solution only if the equation $E(A^T Q A)^{-1} E^T \lambda = d - E(A^T Q A)^{-1} A^T Q b$ has a solution λ .
- e. Assume $k \leq m$ and E has full row rank. Show that the least squares problem under equality constraints has the unique solution:

$$x = (A^T Q A)^{-1} \{A^T Q b + E^T (E(A^T Q A)^{-1} E^T)^{-1} (d - E(A^T Q A)^{-1} A^T Q b)\}.$$

You may use the fact that if a matrix Q is positive definite, also its inverse is positive definite.

- f. Reconsider Example 3.15 to find a Mean-Variance Optimal Portfolio. That is, to find the solution of the optimization problem:

$$\begin{aligned} & \text{minimize}_{\omega_b, \omega_s} \sigma^2(\omega_b, \omega_s) := (1 - \omega_b - \omega_s)^2 \sigma_0^2 + \omega_b^2 \sigma_b^2 + \omega_s^2 \sigma_s^2 \\ & \text{subject to } \mu(\omega_b, \omega_s) := (1 - \omega_b - \omega_s)(1 + r)X + \omega_b(1 + r_b)X + \omega_s(1 + r_s)X = \bar{X}. \end{aligned}$$

Let $x := [\omega_0 \ \omega_b \ \omega_s]$, with $\omega_0 = 1 - \omega_b - \omega_s$. Use above theory to show its solution is: $x = Q^{-1} E^T (E Q^{-1} E^T)^{-1} d$, for appropriately chosen matrices Q and E and vector d . Find the solution if $r = 0.02$, $r_b = 0.05$, $r_s = 0.15$, $X = 100$, $\bar{X} = 105$, $\sigma_0^2 = 0.01$, $\sigma_b^2 = 0.05$ and $\sigma_s^2 = 0.2$.

- g. Consider the unconstrained case, i.e., $E = 0$ and $d = 0$.

Two vectors $y, z \in \mathbb{R}^m$ are called perpendicular if their inner product $y^T z = 0$.

Show that $(A\hat{x})^T Q(b - A\hat{x}) = 0$ if $\hat{x}$ solves (3.12). Interpret this result geometrically, consider first the case $Q = I$.

Problems with an Economic Content

EXERCISE 3.28 (optimal production by a producer)

Consider a producer who wants to minimize his spending on production cost, which consist of labor (L) and capital (K). Assume his cost per unit of labor and capital is ω and r , respectively. His production function is given by the Cob-Douglas function $P(K, L) = \alpha K^\gamma L^{1-\gamma}$ ($\alpha > 0, \gamma \in (0, 1), K > 0, L > 0$).

- Assume the producer wants to produce an amount of goods G . Calculate the amount of labor and capital he should use if he wants to minimize his production cost.
- What will be approximately the additional cost if the producer wants to increase his production by one unit.
- Determine the production of goods at the stationary point of the problem where the producer wants to maximize his profit, wages are flexible and the price he gets on the market for his good is $p = p_0 + \beta\omega - \delta G$.
- Formulate an additional necessary condition for production to occur in this model. What is remarkable about the optimal wage set by the producer? Does this make sense?
- Consider the second-order condition for the stationary point and conclude.
- Calculate the optimum for the profit maximization problem considered in item c, if wages are fixed.
- What happens with the maximal profit if the producer increases wages?

EXERCISE 3.29 (optimal stock acquisition)

Consider a firm that sells two similar products A and B (e.g. iphone's). The amount he can sell from product A, q_A , and B, q_B , is given by, respectively:

$$\begin{aligned} q_A &= -p_A + p_B + 100, \text{ and} \\ q_B &= -2p_B + \frac{3}{2}p_A + 150. \end{aligned}$$

Here p_A and p_B are the prices he asks for product A and B, respectively. The firm has a budget b to spend. The buying-in price for product A is p_1 and for product B is p_2 .

- Assume the firm wants to maximize its sales. Calculate the prices the firm should ask and its optimal stock.
- For which buying-in prices does the firm not make any profit?
- Consider the case that the firm makes profit. Under which conditions of the buying-in prices should the firm try to increase its budget?

EXERCISE 3.30 (derivation of the demand function from a utility function)

Consider a twice differentiable concave utility function $U(x_1, x_2, x_3)$ that is a strictly increasing function for $x_i > 0$, where x_i is the quantity consumed by the consumer of good i . Assume the consumer has a total budget of $m > 0$ to spend and the price of good i is $p_i > 0$.

- a. Show that the consumer's utility is maximal if the marginal rate of substitutions, $\frac{U'_{x_i}}{U'_{x_j}}$, equal the corresponding price relatives.
- b. Consider $U(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. Show that in that case the demand for good i , $d_i(p_1, p_2, p_3) = \frac{\alpha_i}{\sum_{i=1}^3 \alpha_i} \frac{m}{p_i}$.
- c. Calculate by how much maximum utility will change if the budget changes by Δm .
- d. Let $v(p, m) := U(d_1(p, m), d_2(p, m), d_3(p, m))$. Show that $d_i(p, m) = -\frac{v_{p_i}(p, m)}{v_m(p, m)}$ (Roy's identity).

EXERCISE 3.31 (*Engel curves and Income consumption curves*)

Consider a consumer who wants to maximize his utility function $U(x, y, z) \in \mathbb{C}^2$ given his budget I and prices p_i for good i , $i = 1, 2, 3$, from which he consumes a positive amount of x , y and z , respectively.

- a. Use the Lagrange approach to determine the first-order conditions for this maximization problem.
- b. Consider $U(x, y, z) = x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$, where $0 < \alpha_i < 1$, $i = 1, 2, 3$. Determine the optimal amount of consumption of the goods x , y and z as an explicit function of the budget. This are the so-called *Engel curves*.
- c. Show that in the optimal situation, the optimal amount of consumption of product 1, x , is linearly related to both the optimal amount of consumption of product 2, y , and product 3, z , respectively. The functions $x(y)$ and $x(z)$ are called *income consumption curves*.

EXERCISE 3.32 (*deriving the production function for a producer*)

Consider a producer who wants to minimize his spending on production cost, which consist of labor (L) and capital (K). Assume his cost per unit of labor and capital is ω and r , respectively. His production function is given by a concave function $P(K, L)$.

- a. Assume the producer wants to produce an amount of goods y . Show that the amount of labor and capital he should use if he wants to minimize his production cost is determined by the two equations:

$$\begin{aligned} P(K, L) &= y \\ \frac{P'_L}{P'_K} &= \frac{\omega}{r}. \end{aligned}$$

- b. Consider $P(K, L) = \mu(\alpha K^\rho + (1 - \alpha)L^\rho)^{\frac{\beta}{\rho}}$, where $0 < \alpha < 1$, $\mu > 0$, $\beta > 0$ and $\rho < 1$ (a production function that has constant elasticity of substitution, a so-called CES-production function). Show that the with this production function corresponding long run cost function is: $C(y) = \theta y^{\frac{1}{\rho}}$, for some constant θ .
- c. Show, using the second-order criterion that $\det HL < 0$, that the solution obtained in item b indeed yields a minimum. item[d.] Assume the producer will be a monopolist in the market and the price of his product is given by $p(y) = \frac{c}{\sqrt{y}}$ and $0 < \rho < 1$. Calculate the amount of labor and capital the producer should invest to maximize his profit.

EXERCISE 3.33 (*Income and substitution effect in optimal consumer choice if price of a good changes*)

Consider a consumer who wants to spend his budget I on two goods. The price of good i is p_i , $i = 1, 2$. His utility from consumption of amounts x and y of the goods, respectively, is $U(x, y) \in C^2$.

- a. Formulate the optimization problem the consumer has to solve if he wants to maximize his utility.

Use the Lagrange approach to formulate the first order conditions. Assume utility is maximized by a consumption of (x^*, y^*) of the goods.

- b. Assume the price of the first good increases from p_1 to p_1^N . Assume utility is maximized in this case by a consumption of x^N and y^N of both goods, respectively.

Assume the utility function is given by $U(x, y) = xy$. Calculate x^N and y^N .

- c. Next, some authorities want to compensate the consumer for his loss of utility by increasing income such that with these new prices he can still achieve his old utility $U^* := U(x^*, y^*)$. This results in the optimization problem $\max_{x, y} U(x, y)$ subject to the budget constraint $p_1^N x + p_2 y = I^N$ and $U(x, y) = U^*$.

Determine this drop in real income if the utility function is given by $U(x, y) = xy$ and the corresponding optimal consumption bundles (x^S, y^S) . In literature the move from the bundle (x^*, y^*) to the bundle (x^H, y^H) is called the (*Hicks*) *substitution effect* and the move from the bundle (x^H, y^H) to the bundle (x^N, y^N) the (*Hicks*) *income effect*.

- d. Other authorities want to compensate the consumer by increasing income such that he can purchase his old optimal bundle (x^*, y^*) . With this new income, I^S , the consumer can in fact realize a higher utility. Calculate this corresponding maximum utility and consumption bundles (x^S, y^S) if $U(x, y) = xy$. In literature the move from the bundle (x^*, y^*) to the bundle (x^S, y^S) is called the (*Slutsky*) *substitution effect* and the move from the bundle (x^S, y^S) to the bundle (x^N, y^N) the (*Slutsky*) *income effect*.

CHAPTER 4

Constrained Optimization: Inequality Constraints

In this chapter we consider the optimization of a function under inequality constraints which are defined on an open set $C \subset \mathbb{R}^n$. The general formulation of the problem reads as follows.

Problem Statement Let $f, g_i, i = 1, \dots, m$, be functions defined on an open subset $C \subset \mathbb{R}^n$ and $g := [g_1 \cdots g_m]^T$.

$$\text{Maximize } f(s), \text{ subject to } g(s) \leq b, \text{ where } s \in \mathbb{R}^n \text{ and } g : C \rightarrow \mathbb{R}^m. \quad (4.1)$$

Along the lines of the previous chapter let $S = \{x \in \mathbb{R}^n \mid g_i(x) \leq b_i, i = 1, \dots, m\}$. Then the set $D := C \cap S$ will be referred to as the *constraint set* or *feasible region*. Notice that this constraint set is a subset of the domain C and both sets usually do not coincide. Throughout this and ensuing chapters we will assume that this constraint set D is nonempty. For otherwise the constrained optimization problem (4.1) is meaningless.

In literature problem (4.1) is known as the *nonlinear optimization/programming problem*. In case the objective function f is concave and constraint functions g_i are convex functions problem (4.1) is called a *convex optimization problem*. Whereas the subclass of these convex optimization problems where the functions f and g_i are assumed to be linear are called *linear programming problems*. In this chapter, we focus on deriving the necessary conditions for the general nonlinear programming problem (4.1), known as the Karush-Kuhn-Tucker (KKT) necessary conditions. Solving these KKT conditions numerically is generally challenging. In Exercise 4.24, we discuss one numerical approach, the so-called Barrier Method, for solving a constrained optimization problem. However, many other methods have also been proposed in the literature.

For convex, and in particular linear programming problems, highly efficient numerical algorithms have been developed to solve these KKT equations. In the next chapter, we will explore convex and linear programming problems in more detail. Nevertheless, covering all aspects of these problems -

and especially their numerical solution - goes beyond the scope of this book. Readers interested in a deeper theoretical background or in the numerical solution of convex optimization problems (see, e.g., [5]) or linear programming problems (see, e.g., [21], [22], [23]) are encouraged to consult the specialized literature.

As in the case of problems with equality constraints, the formulation in (4.1) allows for the inclusion of multiple constraints. In addition, equality constraints can also be incorporated, as we will demonstrate shortly. However, we first want to emphasize that the KKT necessary conditions, which will be presented later, are specifically designed for the problem as stated in (4.1) - that is, for a maximization problem with " $\leq$ " inequality constraints.

The following example illustrates that many other optimization problems, involving different types of equality or inequality constraints, can be reformulated within the framework of (4.1).

EXAMPLE 4.1

1. Consider the maximization of $f(x, y) = x - y$ subject to the inequality constraints $h_1(x, y) := x \geq 0$, $h_2(x, y) := y \geq 0$, and $g_3(x, y) := x + y \leq 1$.

The first two inequalities can be rewritten as

$$g_1(x, y) := -x \leq 0 \text{ and } g_2(x, y) := -y \leq 0.$$

Using the above notation, define

$$g(x, y) := \begin{bmatrix} g_1(x, y) \\ g_2(x, y) \\ g_3(x, y) \end{bmatrix}.$$

Note that the constraint set is nonempty, even though in this case the number of constraints exceeds the number of variables.

2. Consider the minimization of $\tilde{f}(x, y, z) = x^2 + y^2$ subject to the inequality constraints $g_1(x, y) := x + y \leq 1$, $g_2(x, y) := -x \leq 0$, and $g_3(x, y) := -y \leq 0$.

Instead of minimizing $\tilde{f}$ over this constrained set S , we may equivalently consider the maximization of $f(x, y) = -\tilde{f}(x, y)$ over S .

If the maximum value $\bar{f}$ of $f(x, y)$ on S is attained at $(\bar{x}, \bar{y})$, then the minimum value of $\tilde{f}$ on S equals $-\bar{f}$ and is also attained at $(\bar{x}, \bar{y})$.

3. Consider the maximization of $f(x, y, z) = x - y + z$ subject to the equality constraint $h(x, y, z) := -x + 2y + 3z = 4$ and the inequality constraint $g_3(x, y, z) := x^2 + y^2 + z^2 \leq 9$.

The equality constraint h can be expressed as two inequality constraints that must be satisfied simultaneously, namely $-x + 2y + 3z \geq 4$ and $-x + 2y + 3z \leq 4$.

defining

$$g_1(x, y, z) := -x + 2y + 3z \leq 4 \text{ and } g_2(x, y, z) := x - 2y - 3z \leq 4,$$

the problem can be reformulated within the standard framework (4.1).

EXAMPLE 4.2 Consider a firm that seeks to maximize its profit. The firm's output is described by the production function

$$q = Q(K, L), \text{ where } K \geq 0 \text{ and } L \geq 0,$$

where K and L denote the amounts of capital and labor employed in the production process, respectively. Assume the price of capital is r and the wage rate of labor is ω . The firm faces a budget constraint given by

$$rK + \omega L \leq b,$$

where b represents the maximum budget available. The price of the output good, p , depends on the quantity produced.

This constrained optimization problem can then be reformulated in the standard form, (4.1), as follows.

$$\text{maximize } \pi(K, L) := p(K, L)Q(K, L) - rK - \omega L, \text{ subject to}$$

$$g_1(K, L) := -K \leq 0; \quad g_2(K, L) := -L \leq 0; \quad \text{and} \quad g_3(K, L) := rK + \omega L \leq b.$$

Figure 4.1 provides a geometric illustration of the optimization problem. The goal is to determine the maximum value that the function f attains within the shaded region.

Unfortunately, solving a constrained optimization problem with inequality constraints is generally more complicated than solving one with only equality constraints. Identifying candidate solutions using a first-order criterion requires solving a system of equations that includes both equality and inequality constraints. As noted earlier, finding solutions that satisfy all these constraints is usually a nontrivial task. Moreover, as with the optimization problems considered in previous chapters, points that satisfy the first-order criterion we will derive do not automatically correspond to maximum values of the function. For example, consider the maximization of $f(x) = x^2$ subject to $x \leq 1$. Applying the first-order criterion would yield $x = 0$ as a potential maximum, which is clearly incorrect.

Unless the feasible region is compact, determining whether a global maximum exists often requires additional work. If it is known that a global maximum does exist, it is typically found by directly evaluating the objective function at the candidate solutions. The point(s) at which the function attains its largest value then provide the solution(s) to the constrained optimization problem.

As in the previous chapter, we will first consider the simplest case: a problem with a single inequality constraint and an objective function of two variables. We will discuss both the so-called Kuhn-Tucker conditions that a solution must satisfy and the reasoning used to derive these condi-

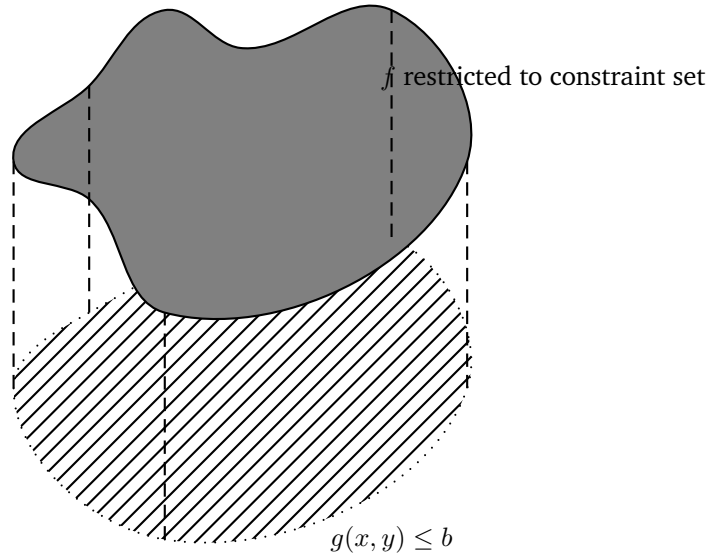


FIGURE 4.1 – *Constrained optimization: inequality constraints.*

tions. In addition, we will examine the second-order condition to determine whether a solution of the Kuhn-Tucker conditions corresponds to a (strict) local maximum. Subsequently, we will extend the discussion to the general case with multiple inequalities and/or more variables.

In deriving the Kuhn-Tucker conditions, we make a regularity assumption. It is also possible to proceed without this assumption, though this requires introducing an additional parameter into the Lagrange function associated with the problem. We will discuss this approach toward the end of the chapter, in what is known in the literature as the Fritz John theorem.

To avoid replacing equality constraints with twice as many inequality constraints, we also present a shortcut for solving optimization problems that involve both equality and inequality constraints. Additionally, we briefly introduce another method for tackling this type of optimization problem, which operates under somewhat less restrictive conditions.

To gain a better understanding of the problem setting, we first consider the concept of the directional derivative in the next section.

4.1 The Directional Derivative

To solve problem (4.1), we need to determine the derivative of a differentiable multi-variable function $f(s)$ in any given direction at a point $s = a$. Let $a = (x_0, y_0)$ be a point in the domain of f . Consider the intersection of the graph of $f(x, y)$ with a vertical plane passing through $s = a$ and oriented in a predetermined direction. Our interest lies in the derivative of this intersection curve at $s = a$.

The orientation of the vertical plane is fully determined by the line along which it intersects the plane

$z = 0$ (assuming $z = f(x, y)$ and the standard coordinate system in three-dimensional space). By choosing an arbitrary vector u along this line, the direction of the vertical plane is fixed. At $s = a$, two directions along the intersection line are possible: if u specifies one direction, then $-u$ points in the opposite direction. Thus, at any point a , the intersection curve has two directional derivatives that are equal in magnitude but opposite in sign. Figure 4.2 illustrates this situation.

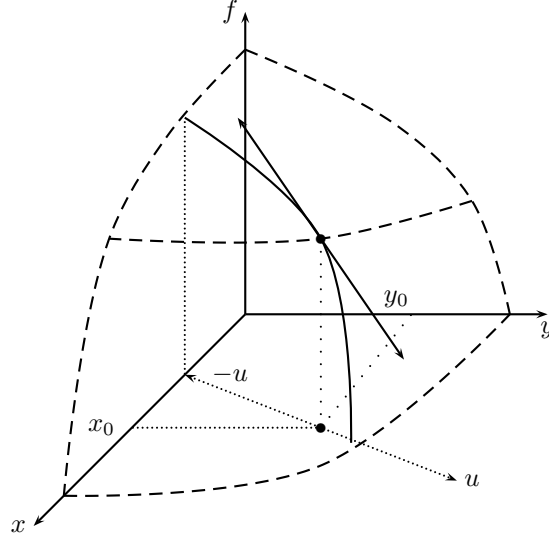


FIGURE 4.2 – Directional derivative at (x_0, y_0) in direction u .

More formally, let f be a differentiable function. Then, at any point $s = a$, we can calculate the derivative of the intersection function of f with a vertical plane in the direction of a vector u . We call this the *directional derivative of f at $s = a$ in the direction u* . Special cases are the partial derivatives. For instance, $D_1 f(a)$ is the directional derivative of f in $s = a$ in the direction $u = [1 \ 0 \ \dots \ 0]^T$. The directional derivative of f at a point $s = a$ in the direction u represents the slope of the tangent line to the graph of f along that direction. In other words, it indicates approximately how much f will change if we move a in the direction u . Thus, in particular, the directional derivatives at $s = a$ provide information about the direction in which $f(s)$ increases most rapidly. Calculating the directional derivative in a given direction is relatively straightforward. We have the following result (see also Exercise 4.21, where the reader is asked to provide a formal proof).

PROPOSITION 4.3 *Let $f(s)$ be differentiable on an open set $S \subset \mathbb{R}^n$ and $a \in S$. Then, f is differentiable at a in the direction u . Furthermore, the directional derivative of f at a in the direction u , denoted by $D_u f(a)$, equals*

$$D_u f(a) = Df(a) * \frac{u}{\|u\|}.$$

Here, $\|u\|$ denotes the length of the vector $u = [u_1, \dots, u_n]^T$: $\sqrt{u_1^2 + \dots + u_n^2}$.

EXAMPLE 4.4 Consider $f(x, y) = x^2 + y^2$. Then, the directional derivative of f at (x, y) in the direction $u = [3 \ 4]^T$ is:

$$D_u f(x, y) = [2x \ 2y] * \frac{1}{\sqrt{3^2 + 4^2}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{5}(6x + 8y).$$

So, for example, $D_u f(5, 10) = 22$.

Next, we consider the question of which directions, at a fixed point a in the domain, cause a function f to increase minimally and which directions produce the maximal increase. Clearly, the function's value changes very little in those directions u for which $D_u f(a) = 0$. In the case where f is a function of two variables, Proposition 4.3 shows that these directions are given by the vectors $u = [u_1 \ u_2]^T$ for which

$$Df(a) * u = [D_1 f(a) \ D_2 f(a)] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = D_1 f(a)u_1 + D_2 f(a)u_2 = 0, \text{ i.e., } u = \begin{bmatrix} -D_2 f(a) \\ D_1 f(a) \end{bmatrix}.$$

Geometrically, the directions in which the function value at $s = a$ remains nearly constant are exactly those vectors perpendicular to $(Df(a))^T$. Figure 4.3, panel (a), illustrates those directions.

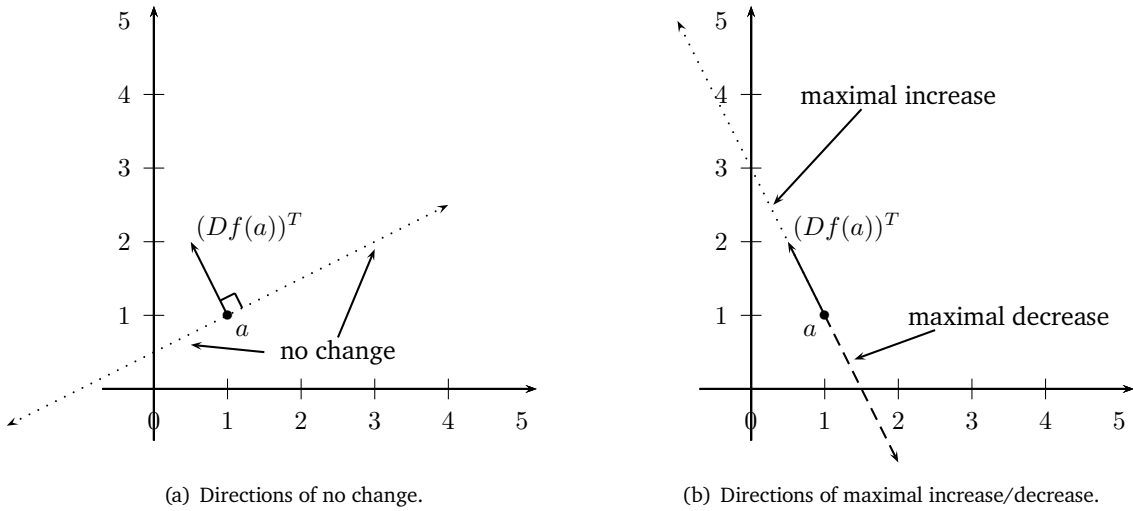


FIGURE 4.3 – Directions in which f changes nothing/most.

Determining the vectors u along which the function value changes most rapidly requires a bit more work. To this end, we need a general property that provides an estimate for the product of a row and a column vector, that is: if $a, b \in \mathbb{R}^n$, then (see Exercise 2.35)

$$|a^T b| \leq \|a\| \|b\| \quad (\text{inequality of Cauchy – Schwartz}). \quad (4.2)$$

For example, if $a = [1 \ 2]^T$ and $b = [3 \ 4]^T$, the above inequality (4.2) states that $3 + 8 \leq \sqrt{5} * 5$. Using (4.2), we can estimate the absolute value of the directional derivative as follows:

$$|D_u f(a)| = |Df(a) * \frac{u}{\|u\|}| = \frac{1}{\|u\|} |Df(a) * u| \leq \frac{1}{\|u\|} \|(Df(a))^T\| \|u\| = \|(Df(a))^T\|. \quad (4.3)$$

On the other hand, if we choose the direction $u = (Df(a))^T$, then

$$D_u f(a) = Df(a) * \frac{(Df(a))^T}{\|(Df(a))^T\|} = \|(Df(a))^T\|^2 * \frac{1}{\|(Df(a))^T\|} = \|(Df(a))^T\|. \quad (4.4)$$

From (4.3) and (4.4) it follows that the directional derivative is maximized when we choose the direction $u = (Df(a))^T$. Conversely, the function *decreases* most rapidly in the direction $u = -(Df(a))^T$. These directions are illustrated in Figure 4.3, panel (b). We formalize these observations in the following result.

PROPOSITION 4.5 Assume $f(s)$ is differentiable in $s = a$. Then,

- The value of f increases maximally in the direction $u = (Df(a))^T$.
 - The value of f decreases maximally in the direction $u = -(Df(a))^T$.
 - The value of f remains almost constant in every direction that is perpendicular to $u = (Df(a))^T$.
- The direction $u = (Df(a))^T$ is called the *gradient* of f in $s = a$.

EXAMPLE 4.6 Reconsider the function $f(x, y) = x^2 + y^2$. Its derivative is $Df(x, y) = [2x \ 2y]$. Hence, $Df(1, 2) = [2 \ 4]$.

Therefore, by Proposition 4.5, the value of f increases maximally in the direction $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$; the value of f decreases maximally in the opposite direction $u = -\begin{bmatrix} 2 \\ 4 \end{bmatrix}$; and f remains almost constant in the directions of $u = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $u = -\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, respectively.

EXERCISE 4.1 Calculate in which directions the function f at the indicated point increases maximally, minimally and almost not, respectively.

- a. $f(x, y) = 4xy$ at $(x, y) = (1, 2)$.
- b. $f(x, y) = x^2 + y^2$ at $(x, y) = (0, 0)$.
- c. $f(x, y, z) = x^2 - 2y^2 + z^2$ at $(x, y, z) = (2, 1, 2)$.

Now, consider the constraint set defined by $D := \{(x, y) \mid g(x, y) \leq b\}$. Consider an arbitrary element on the boundary of D , i.e., a point (x_0, y_0) where $g(x_0, y_0) = b$.

By Proposition 4.5, the vector $(Dg(x_0, y_0))^T$ points in the direction in which $g(x, y)$ increases most rapidly. This implies, that $(Dg(x_0, y_0))^T$ points in a direction that is not contained in D , since all

values in D are less than or equal to b . Because the values of g remain constant along the boundary, Proposition 4.5 further implies that $g(x, y)$ increases most in the direction perpendicular to the tangent line of the graph of g at (x_0, y_0) . This situation is illustrated in Figure 4.4.

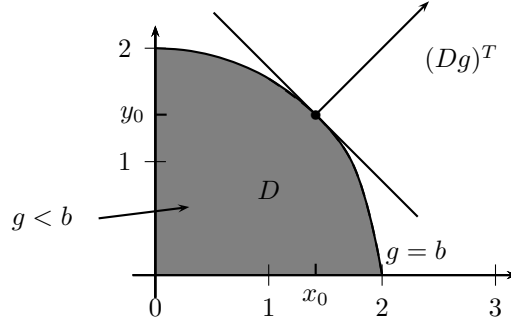


FIGURE 4.4 – Direction of maximal increase of g on the boundary set

More generally, we have the following result. The reader is asked to provide a proof in Exercise 4.15.

LEMMA 4.7 Assume $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous differentiable on an open set $S \subset \mathbb{R}^n$ containing a . Consider the set $C = \{x \in \mathbb{R}^n \mid g(x) \leq b\}$ and assume $g(a) = b$. Then,

1. $Dg_u(a)u \geq 0$ for any direction u pointing into C .
2. $Dg_u(a)u \leq 0$ for any direction u pointing outside C .
3. $Dg(a)$ is perpendicular to the tangent plane of the graph of $g(x)$ at $x = a$.

We will use this property in the next sections to derive first-order and second-order conditions for the constrained optimization problem.

4.2 Two variables and one inequality constraint

In this section we consider the next constrained optimization problem.

Problem Statement Let $C \subset \mathbb{R}^2$ be an open set.

$$\text{Maximize } f(x, y), \text{ subject to } g(x, y) \leq b, \text{ where } f, g : C \rightarrow \mathbb{R}. \quad (4.5)$$

As in Chapter 3.2, we employ a geometric approach to solve the problem. As an example consider the problem to find the maximum of $f(x, y)$ in case f and the constraint set are given by:

$$f(x, y) = xy \text{ and } g(x, y) := x^2 + y^2 \leq 4.$$

For simplicity, let us further assume that we are only interested in the case where both $x > 0$ and $y > 0$. Figure 4.5 shows the feasible region, indicated by the shaded area. Suppose, for the moment, that this problem has a solution at $(x, y) = (c, d)$.

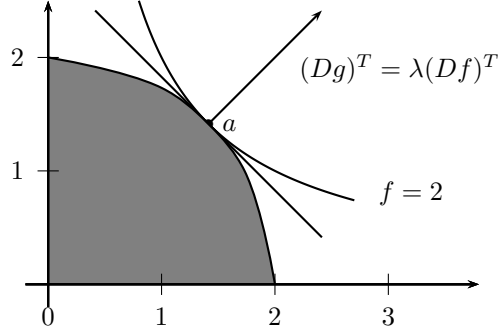


FIGURE 4.5 – Optimal allocation in Problem 4.5.

Depending on its location, the point (c, d) can fall into one of two cases.

1. Either, it is an interior point of the feasible region, i.e., $g(c, d) < b$, or,
2. it is a boundary point of the feasible region, i.e., $g(c, d) = b$. In that case we say that the constraint is *binding* (or *tight* or *active*).

In case (c, d) is an interior point of the feasible region, (c, d) is (at least) a local maximum of $f(x, y)$ in some open set around (c, d) . That is, (c, d) solves an unconstrained optimization problem. So, the first-order conditions for an unconstrained optimum apply. That is, by Theorem 2.32, $Df(c, d) = [0 \ 0]$. By choosing the scalar $\lambda = 0$, below, we see then that the next equalities hold:

$$Df(c, d) - \lambda Dg(c, d) = 0, \text{ for some } \lambda \geq 0, \quad (4.6)$$

$$\lambda(g(c, d) - b) = 0. \quad (4.7)$$

Since at least one of the constraints $\lambda \geq 0$ or $g(c, d) - b \leq 0$ must be binding (as expressed in (4.7)), condition (4.7) is referred to as a *complementary slackness* condition.

In case (c, d) is a boundary point, it is also a maximum for $f(x, y)$ on the constraint set $g(x, y) = b$. So, if $Dg(c, d) \neq [0 \ 0]$, there is a λ such that (c, d) is a stationary point of the Lagrange function $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$. That is, $Df(c, d) = \lambda Dg(c, d)$. Or, by taking the transpose, $(Df(c, d))^T = \lambda(Dg(c, d))^T$. Stated differently, the gradient of f is a multiple of the gradient of g in this case. We consider this equality in some more detail.

As noted in the previous subsection, the gradient of g points outward from the constraint set along its boundary. Now, consider the gradient of f . If the gradient of f were pointing inward, Proposition 4.5 would imply that f could attain higher values inside the constraint set. This, however, contradicts our assumption that f reaches its maximum on the boundary. In other words, the gradient of f also points outward. Figure 4.5 illustrates this situation. From the relationship $Df(c, d) = \lambda Dg(c, d)$ we

therefore conclude that $\lambda \geq 0$, so (4.6) holds. Moreover, since g is binding, (4.7) also holds.

In summary, we have shown that, under certain regularity assumptions, whether the maximum occurs in the interior or on the boundary of the constraint set, equations (4.6) and (4.7) are satisfied. This leads to the following theorem.

THEOREM 4.8 (Kuhn-Tucker)

Consider the constrained maximization problem (4.5). Assume both $f, g \in C^1$. Assume the problem has a solution at (c, d) and that, in case the constraint is binding (i.e. $g(c, d) = b$), $Dg(c, d) \neq [0 \ 0]$. Let $L(x, y, \lambda) := f(x, y) - \lambda(g(x, y) - b)$ be the with this problem associated Lagrange function. Then, there exists a $\lambda \in \mathbb{R}$ such, that (c, d, λ) satisfies the next set of (in)equalities

1. $[D_1 L(c, d, \lambda) \ D_2 L(c, d, \lambda)] = [0 \ 0]$;
 2. $\lambda(g(c, d) - b) = 0$;
 3. $g(c, d) \leq b$ and $\lambda \geq 0$.
- (KKT)

From Theorem 4.8 we deduce the next algorithm to solve our constrained maximization problem (4.5).

ALGORITHM 4.9 To find the candidate solutions of our constrained maximization problem (4.5), proceed as follows:

Step 1. Verify whether $f, g \in C^1$.

Step 2. Find all points where simultaneously $Dg = [0 \ 0]$ and $g(x, y) = b$.

Step 3. Write down the Lagrange function

$$L(x, y, \lambda) := f(x, y) - \lambda(g(x, y) - b).$$

Step 4. Determine all solutions (x, y, λ) that satisfy conditions 1, 2 and 3 from Theorem 4.8.

All points found in Step 2 and Step 4 above, are candidate solutions for the constrained optimization problem (4.5).

In the algorithm above, Step 2 is included because at points in the feasible region where the derivative is zero, the Implicit Function Theorem does not apply. Since the theory we have developed relies on the assumption that the Implicit Function Theorem holds, we cannot make conclusions at points where this assumption fails. In particular, it is possible that the function attains an optimum at such points.

If we know in advance that the constrained maximization problem 4.5 has a solution, then at least one of the candidate solutions identified by Algorithm 4.9 must solve the problem. When there are only a finite number of candidates, simply evaluating the corresponding function values allows us to identify the solution: the candidate at which the function attains its largest value provides the maximum we are looking for.

However, if the constraint set is not compact, determining whether a solution exists is generally not straightforward. The following example illustrates a case in which the constrained maximization problem has no solution, even though the algorithm produces candidate solutions.

EXAMPLE 4.10 Consider the maximization of $f(x, y) = -xy$ on the constraint set $g(x, y) := x + y \leq 2$. By considering $(x, y) = (-a, 1)$ with a arbitrarily large, the constraint is satisfied while

$$f(-a, 1) = a$$

becomes arbitrarily large. Hence, the problem has no global maximum.

Next, consider Algorithm 4.9.

Step 1. Clearly $f, g \in C^1$.

Step 2. $Dg = [1 \ 1]$, which is clearly full row rank. So, we don't get any potential candidate solutions from this step.

Step 3. The with this problem corresponding Lagrangian is: $L(x, y, \lambda) = -xy - \lambda(x + y - 2)$.

Step 4. The necessary conditions are:

1. $[-y - \lambda \ -x - \lambda] = [0 \ 0]$
2. $\lambda(x + y - 2) = 0$
3. $x + y \leq 2$ and $\lambda \geq 0$

From 1 we have $x = y = -\lambda$. From 2 we have either $\lambda = 0$ or $x + y = 2$. It is easily verified that $(x, y, \lambda) = (0, 0, 0)$ satisfies all 3 conditions. Furthermore, we have that if $x + y = 2$ holds necessarily $(x, y) = (1, 1)$ applies. But then $\lambda = -1 < 0$, which violates condition 3.

So, the algorithm does provide a candidate solution: $(x, y) = (0, 0)$.

By considering $(x, y) = (-\epsilon, \epsilon)$, for ϵ close to zero, one verifies that the constrained maximization problem doesn't have a local maximum at the candidate solution $(0, 0)$. In fact f has a saddle point at $(0, 0)$.

The example above highlights the need for criteria that allow us to determine whether a constrained maximization problem has a solution.

If the constraint set is compact, Weierstrass' theorem guarantees the existence of a global maximum. However, when the constraint set is not compact, as the example shows, determining the existence of a maximum becomes more complicated.

There is, however, another sufficient condition that is easy to verify and ensures that a candidate solution is a global maximum. That is, in case the Lagrange function is concave at the stationary point. Proposition 4.11, below, states the precise result.

PROPOSITION 4.11 *Let f and g be differentiable on an open convex set C . Assume (c, d, λ_0) are a solution of the set of Kuhn-Tucker (in)equalities 1-3 from Theorem 4.8. Then, if $\tilde{L}(x, y) := L(x, y, \lambda_0)$ is concave on C , the constrained maximization problem (4.5) has a global maximum at (c, d) .*

Verifying the correctness of the above statement is straightforward. Since the Lagrange function has a stationary point at (c, d, λ_0) , the function $\tilde{L}(x, y)$ has a stationary point at (c, d) . By assumption, $\tilde{L}(x, y)$ is concave, so Corollary 2.38 implies that $\tilde{L}(x, y)$ attains a global maximum at (c, d) ; that is, $\tilde{L}(x, y) \leq \tilde{L}(c, d)$ for all $(x, y) \in C$. Using this, we conclude that

$$\begin{aligned} f(x, y) - f(c, d) &= f(x, y) - \lambda_0(g(x, y) - b) - [f(c, d) - \lambda_0(g(x, y) - b)] \\ &= \tilde{L}(x, y) - f(c, d) + \lambda_0[g(x, y) - b] \\ &\leq \tilde{L}(c, d) - f(c, d) + \lambda_0[g(x, y) - b] \\ &= \lambda_0(g(c, d) - b) + \lambda_0[g(x, y) - b] = \lambda_0[g(x, y) - b]. \end{aligned}$$

However, both λ_0 and $b - g(x, y)$ are nonnegative. Therefore, in the last equation above, $\lambda_0[g(x, y) - b] \leq 0$. From this we conclude that $f(x, y) \leq f(c, d)$ for all (x, y) in the constraint set. In other words, the objective function attains a global maximum at (c, d) .

If the two criteria above do not apply, one can assess whether a maximum exists by examining the second-order derivative of f when (c, d) is an interior point of the constraint set. If (c, d) lies on the boundary, a second-order criterion - similar to Theorem 3.38 (see Theorem 4.19 below for details) - can be used to determine whether a maximum is attained.

EXAMPLE 4.12 Consider the constrained maximization problem

$$\text{maximize } f(x, y) = 2(x + 1)^2 + y^2 - 2 \text{ subject to } g(x, y) := x^2 + y^2 \leq 4.$$

First note that $f(x, y)$ is continuous and the constraint set is compact. So, by Weierstrass' theorem, this problem has a global maximum.

To find this maximum, we proceed along the lines of Algorithm 4.9.

Step 1. $Df(x, y) = [4x + 4 \ 2y]$ and $Dg(x, y) = [2x \ 2y]$. Clearly, all partial derivatives are continuous.

So, $f, g \in C^1$.

Step 2. $Dg(x, y) = [0 \ 0]$ only in case $(x, y) = (0, 0)$. But at $(0, 0)$, $g(0, 0) = 0 \neq 4$. So, the constraint is not binding in $(0, 0)$. Therefore, we don't find any candidate solutions in this step.

Step 3. The Lagrange function is: $L(x, y, \lambda) = 2x^2 + y^2 + 4x - \lambda(x^2 + y^2 - 4)$.

Step 4. The necessary conditions are:

1. $[4x + 4 - 2\lambda x \ 2y - 2\lambda y] = [0 \ 0]$.
2. $\lambda(x^2 + y^2 - 4) = 0$.
3. $x^2 + y^2 \leq 4$ and $\lambda \geq 0$.

From 2 we have that either $\lambda = 0$ or $x^2 + y^2 = 4$. We consider these cases consecutively.

Case 1: $\lambda = 0$.

By 1, $x = -1$ and $y = 0$. It is easily verified that $(x, y, \lambda) = (-1, 0, 0)$ satisfies 3 too. So, $(-1, 0, 0)$ is a candidate solution.

Case 2: $x^2 + y^2 = 4$ and $\lambda > 0$.

By 1, $2y(1 - \lambda) = 0$. So, either $y = 0$ or $\lambda = 1$. We consider both cases consecutively.

Case 2.a: $y = 0$.

Then, we have $x^2 = 4$. That is, either $x = 2$ or $x = -2$. If $x = 2$, by 1, $\lambda = 3$. So, $(x, y, \lambda) = (2, 0, 3)$ is a candidate solution.

If $x = -2$, by 1, $\lambda = 1$. So, $(x, y, \lambda) = (-2, 0, 1)$ is another candidate solution.

Case 2.b: $\lambda = 1$.

Then, by 1, we have that this case does not provide additional candidate solutions.

To see where the maximum is attained, we evaluate f at the candidate solutions.

$f(-1, 0) = -2$, $f(2, 0) = 16$ and $f(-2, 0) = 0$. Hence, by Weierstrass' theorem, the maximum is attained at $(2, 0)$.

EXAMPLE 4.13 Consider the constrained maximization problem

$$\text{maximize } f(x, y) = -(x - 2)^2 - (y - 1)^2 + 10 \text{ subject to } g(x, y) := x + y \leq 1.$$

First note that the constraint set is not compact. So, we cannot use Weierstrass' theorem to conclude whether this problem has a global maximum.

To find candidates for a maximum location we proceed along the lines of Algorithm 4.9.

Step 1. $Df(x, y) = [-2x + 4 \quad -2y + 2]$ and $Dg(x, y) = [1 \ 1]$. Clearly, all partial derivatives are continuous. So, $f, g \in C^1$.

Step 2. $Dg(x, y) = [1 \ 1]$. Clearly, $Dg(x, y)$ is full row rank everywhere. So, we don't find any candidate solutions in this step.

Step 3. The Lagrange function is: $L(x, y, \lambda) = -x^2 - y^2 + 4x + 2y + 5 - \lambda(x + y - 1)$.

Step 4. The necessary conditions are:

1. $[-2x + 4 - \lambda \quad -2y + 2 - \lambda] = [0 \ 0]$.
2. $\lambda(x + y - 1) = 0$.
3. $x + y \leq 1$ and $\lambda \geq 0$.

From 2 we have that either $\lambda = 0$ or $x + y = 1$. We consider these cases consecutively.

Case 1: $\lambda = 0$.

By 1, $x = 2$ and $y = 1$. But in that case $x + y = 3 > 1$, which violates 3.

Case 2: $x + y = 1$ and $\lambda > 0$.

So, $y = 1 - x$. Substitution of this into 1, $D_2L = 0$, yields: $\lambda = 2x$. Substitution of this into $D_1L = 0$ gives $-4x + 4 = 0$. Consequently, $x = 1$, $y = 0$ and $\lambda = 2$ is the only candidate solution.

To determine whether this candidate $(x, y, \lambda) = (1, 0, 2)$ yields a global maximum, we consider the Lagrange function at the stationary point:

$$\tilde{L}(x, y) = -x^2 - y^2 + 4x + 2y + 5 - 2(x + y - 1) = -x^2 - y^2 + 2x + 7.$$

Its derivative is

$$\tilde{L}'(x, y) = [-2x + 2 \quad -2y]$$

and the Hessian is

$$H\tilde{L}(x, y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

It is easily verified that $-H\tilde{L}(x, y)$ is positive definite. Consequently, $H\tilde{L}(x, y)$ is negative definite, so $\tilde{L}(x, y)$ is concave. By Proposition 4.11, this implies that $f(1, 0) = 8$ is the global maximum of the problem.

EXAMPLE 4.14 Consider a consumer who wants to maximize his utility, $U(x, y)$, where x and y is the amount he buys from products one and two, respectively. Assume the price of product i is $p_i > 0$, $i = 1, 2$, and that he has a maximum budget of $I > 0$ euros to spend on buying these goods. This problem can be rewritten as the next constrained maximization problem:

$$\text{maximize } U(x, y) \text{ subject to } g(x, y) := p_1x + p_2y \leq I.$$

We assume that the utility function satisfies the following properties. First, utility is strictly increasing in both of its arguments: that is, consuming more of either product increases utility, so $D_iU > 0$, $i = 1, 2$. Second, we assume that utility is concave in its arguments, meaning that the Hessian matrix of U , HU , is negative definite. In other words, the more one consumes, the smaller the additional utility gained.

Note that the constraint set is not compact, so we cannot rely on Weierstrass' theorem to guarantee the existence of a global maximum.

To identify candidate points for the maximum, we follow the procedure outlined in Algorithm 4.9.

Step 1. $DU = [D_1U \ D_2U]$ and $Dg = [p_1 \ p_2]$. Clearly, all partial derivatives are continuous. So, $U, g \in C^1$.

Step 2. $Dg(x, y) = [p_1 \ p_2]$. Clearly, Dg is full row rank everywhere. So, we don't find any candidate solutions in this step.

Step 3. The Lagrange function is: $L(x, y, \lambda) = U(x, y) - \lambda(p_1x + p_2y - I)$.

Step 4. The necessary conditions are:

1. $[D_1U - \lambda p_1 \ D_2U - \lambda p_2] = [0 \ 0]$.
2. $\lambda(p_1x + p_2y - I) = 0$.
3. $p_1x + p_2y \leq I$ and $\lambda \geq 0$.

From 2 we have that either $\lambda = 0$ or $p_1x + p_2y = I$. We consider these cases consecutively.

Case 1: $\lambda = 0$.

By 1, $D_iU = 0$. However, since by assumption $D_iU > 0$, this case doesn't provide a candidate

soluton.

Case 2: $p_1x + p_2y = I$ and $\lambda > 0$.

A simple rewriting of condition 1 yields then: $\frac{D_1U}{D_2U} = \frac{p_1}{p_2}$, with $\lambda^* = \frac{D_iU}{p_i} > 0$.

Without giving a proof we note that, from our assumption on U , it follows that the budget constraint and the equation $\frac{D_1U}{D_2U} = \frac{p_1}{p_2}$ determine x and y uniquely as a function of prices and income.

To determine whether this candidate corresponds to a global maximum, we examine the Lagrange function at the stationary point:

$$\tilde{L}(x, y) = U(x, y) - \lambda^*(p_1x + p_2y - I).$$

It is easily verified that $H\tilde{L}(x, y) = HU(x, y)$. Since, by assumption, $HU(x, y)$ is negative definite, $H\tilde{L}(x, y)$ is negative definite everywhere too. That is, $\tilde{L}(x, y)$ is concave. Therefore, by Proposition 4.11, the candidate solution yields a global maximum of the problem.

EXERCISE 4.2 Solve the next constrained optimization problems.

- Maximize $f(x, y) = 4xy$ subject to the constraint $g(x, y) := x^2 + 2y^2 \leq 6$.
- Maximize $f(x, y, z) = -x^2 - y^2 + 4x + 4y$, subject to the constraint $g(x, y) := x + y \leq 2$.
- Minimize $f(x, y) = x^2 + y^2$, subject to the constraint $g(x, y) := p_x x + p_y y \leq 1$, where p_x and p_y are some positive numbers.
- Minimize $f(x, y) = x + y$, subject to the constraint $g(x, y) := x^2 + y^2 \geq 8$.

4.3 The general case

In this section, we address the general problem (4.1), where the optimization involves more than two variables, multiple constraints, or both. We first recall the problem under consideration.

Problem Statement

Let $C \subset \mathbb{R}^n$ be an open set.

$$\text{Maximize } f(s), \text{ subject to } g_i(s) \leq b_i, \text{ where } f, g_i : C \rightarrow \mathbb{R}, i = 1, \dots, m. \quad (4.8)$$

With $g(s) := [g_1(s) \cdots g_m(s)]^T$ we have the next generalization of Theorem 4.8.

THEOREM 4.15 (Kuhn-Tucker)

Consider the constrained maximization problem (4.8). Assume both $f, g \in C^1$. Assume the problem has a solution at $s = a$ and in this point $k < n$ constraints are binding. Let

$$L(s, \lambda_1, \dots, \lambda_m) := f(s) - \lambda_1(g_1(s) - b_1) - \cdots - \lambda_m(g_m(s) - b_m)$$

be the with this problem associated Lagrange function. Then, if at $s = a$ the derivatives of the set of constraints that are binding are linearly independent¹¹, there exist $\lambda_i \in \mathbb{R}$, $i = 1, \dots, m$, such that $(a, \lambda_1, \dots, \lambda_m)$ satisfy the next set of (in)equalities

1. $[D_1 L(a, \lambda_1, \dots, \lambda_m) \ \cdots \ D_n L(a, \lambda_1, \dots, \lambda_m)] = [0 \ \cdots \ 0]$.
2. $\lambda_i(g_i(a) - b_i) = 0$, $i = 1, \dots, m$. (KKT)
3. $g_i(a) \leq b_i$ and $\lambda_i \geq 0$, $i = 1, \dots, m$.

Condition 1 in the theorem above can be rewritten as $Df^T(a) = \sum_{i=1}^m \lambda_i Dg_i^T(a)$, with $\lambda_i \geq 0$. Geometrically, this means that for f to attain a maximum on the constraint set, the gradient of f at the point a must be a non-negative linear combination of the gradients of the constraint functions at that point. Equivalently, in algebraic terms, the gradient of f at a lies in the convex cone spanned by the gradients of the constraint functions g_i at a .

REMARK 4.16 Recall from the previous Chapter that if the condition in Theorem 4.15 holds - namely, that at the maximum $s = a$, the matrix of derivatives of the set of binding constraints has full row rank - then a is called a *regular point*. In the literature, this condition is referred to as the *linear independence constraint qualification*. This condition can be expressed in an alternative, and less cumbersome, way. To this end, let $\text{diag}(g_i)$ denote the square $m \times m$ diagonal matrix whose i^{th} diagonal entry is g_i . The above condition can then be reformulated as requiring that the matrix G , defined below, has full row rank m for all s in the constraint set.

$$G = [Dg(s) \mid \text{diag}(g_i(s) - b_i)]. \quad (4.9)$$

For example, consider the constraint set $\Gamma = \{(x, y) \mid g_1(x, y) := x^2 + y^2 - 1 \leq 0, g_2(x, y) := -x \leq 0, g_3(x, y) := -y \leq 0\}$. Then the condition in Theorem 4.15 - that at the maximum the derivative matrix of the binding constraints has full row rank - can be replaced by the assumption that the matrix

$$G = \begin{bmatrix} 2x & 2y & x^2 + y^2 - 1 & 0 & 0 \\ -1 & 0 & 0 & -x & 0 \\ 0 & -1 & 0 & 0 & -y \end{bmatrix}$$

has rank 3 for all $(x, y) \in \Gamma$.

REMARK 4.17 The assumption in Theorem 4.15 that, at the maximum point $s = a$, the derivatives of the binding constraints are linearly independent can be relaxed. This relaxation, however, comes at the cost of introducing a so-called parameterizability condition, which is not easy to verify. For further details, see, for example, Takayama [28], Section 1.D.

¹¹ Note this is equivalent with the statement that the matrix consisting of the derivatives of the set of constraints that are binding has full row rank.

Moreover, by imposing additional assumptions on the constraint set or the constraint functions, the need to verify this parameterizability condition can be eliminated. This is the case in particular for convex optimization problems, which will be discussed in the next chapter.

The proof of Theorem 4.15 follows along the same lines as the arguments used to establish Theorem 4.8.

Before presenting a formal proof of this theorem, we first provide some geometric intuition for the case in which two constraints are binding. Figure 4.6 illustrates this situation.

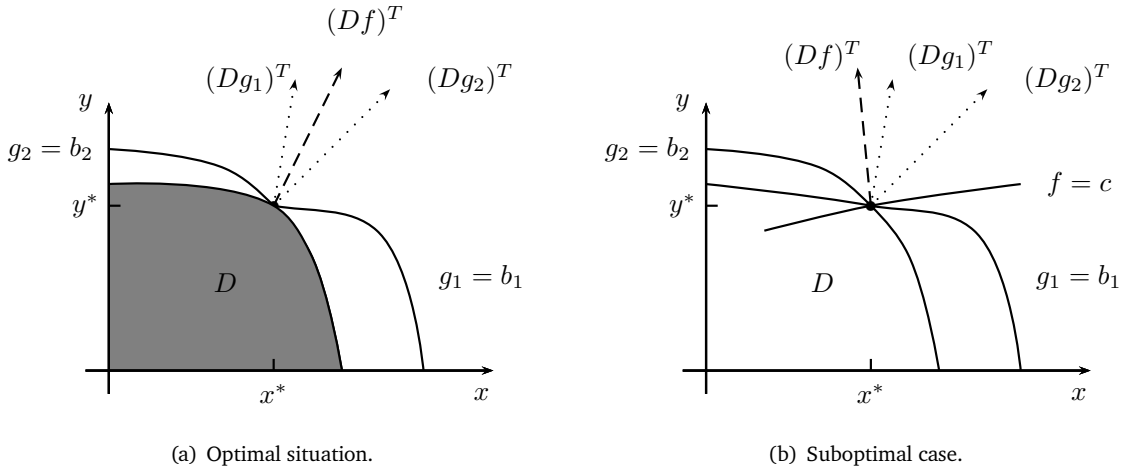


FIGURE 4.6 – Characterization of optimal allocation under two binding constraints.

Panel (a) illustrates the optimal situation. The maximum is attained at the intersection point (x^*, y^*) of the curves $g_1(x, y) = b_1$ and $g_2(x, y) = b_2$. Also shown are the gradients of g_1 and g_2 at this point, which, by Proposition 4.5, are perpendicular to the tangent lines of the corresponding graphs and point outward from the feasible set D .

For the same reasons as in the case where only one constraint is binding, the gradient of f must also point outward from D . In this case, however, we can be more precise about its location: it must lie in the region spanned by the gradients of g_1 and g_2 .

Panel (b) illustrates why this is necessary. In this panel, the gradient of f is shown to lie to the left of the gradient of g_1 . The figure demonstrates that, in this situation, the level curve of f passing through (x^*, y^*) intersects the feasible region D to the left of (x^*, y^*) . Consequently, there exist higher level values of f whose level curves still intersect the feasible region D .

Thus, a higher value of the objective function can be achieved within D , implying that (x^*, y^*) cannot be optimal. By the same reasoning, the gradient of f cannot lie to the right of the gradient of g_2 . Since the gradient of f is enclosed by the gradients of the binding constraints, it follows that the gradient of f can be expressed as a positive linear combination of the gradients Dg_1 and Dg_2 at the maximum point.

A more formal justification of the theorem proceeds as follows. Suppose that, at the optimum, constraint i is nonbinding, and set $\lambda_i^* = 0$. The remaining constraints then define a constrained optimization problem with equality constraints that satisfies all the assumptions of Theorem 3.32. Let λ_j^* denote the Lagrange multipliers associated with these constraints, as implied by Theorem 3.32. With this choice, Conditions 1 and 2 of Theorem 4.15 are clearly satisfied. It remains to show that $\lambda_i^* \geq 0$.

Assume, to the contrary, that $\lambda_i < 0$. For simplicity, suppose that the additional assumptions of Corollary 3.36 hold, so that $\frac{df(s)}{db_i} = \lambda_i$ at $s = a$. This implies that the maximal value of f decreases when b_i is increased. However, increasing b_i enlarges the constraint set, and therefore the maximum value of f on this expanded set must be at least $f(a)$. This contradiction shows that the assumption $\lambda_i < 0$ is false. Hence, $\lambda_i^* \geq 0$.

To determine whether a maximum is attained at a point satisfying the Kuhn–Tucker conditions, we can proceed in the same manner as in the previous section. If the constraint set is compact, Weierstrass' theorem guarantees the existence of a global maximum. When the constraint set is not compact, we can rely either on the following generalization of Proposition 4.11 or on Theorem 4.19 to decide whether a maximum is attained. Note that Theorem 4.19 provides only a sufficient condition for the existence of a (strict) local maximum at the stationary point.

PROPOSITION 4.18 *Let f and g be differentiable on an open convex set C . Assume (a, λ_0) are a solution of the set of Kuhn-Tucker (in)equalities 1-3 from Theorem 4.15. Then, if $\tilde{L}(x) := L(x, \lambda_0)$ is concave on C , the constrained maximization problem (4.8) has a global maximum in $s = a$.*

THEOREM 4.19 *(sufficient/necessary second-order condition for a constrained optimum)*

Consider the constrained optimization problem (4.8) with $f, g =: [g_1^T \ g_2^T]^T \in C^2$, where $g_1 \in \mathbb{R}^{m_1}$, $m_1 < n$, and $g_2 \in \mathbb{R}^{m-m_1}$.

Assume the (KKT) conditions are met at $(a, \lambda_1, \dots, \lambda_{m_1}, 0, \dots, 0)$, where the constraints g_1 are binding and the constraints g_2 are non-binding. Assume $g'_1(a)$ has full row rank. Let $g'_1(a) =: [D_x g_1(a) \ D_y g_1(a)]$, where $D_y g_1(a)$ is the last $m_1 \times m_1$ block of matrix $g'_1(a)$. Assume, without loss of generality, that $D_y g_1(a)$ is invertible. Let

$$H\tilde{L} := Hf(a) - \sum_{i=1}^{m_1} \lambda_i Hg_i(a) \text{ and } B := \begin{bmatrix} I_{(n-m_1) \times (n-m_1)} \\ -(D_y g_1(a))^{-1} * D_x g_1(a) \end{bmatrix}.$$

Then, $f(a)$ will be a local maximum of the constrained optimization problem if $B^T * H\tilde{L} * B$ is negative definite. In case $B^T * H\tilde{L} * B$ is not negative semi-definite the constrained optimization problem has not a local maximum at $x = a$. Here $I_{(n-m_1) \times (n-m_1)}$ denotes the identity matrix in $\mathbb{R}^{(n-m_1) \times (n-m_1)}$.

Theorem 4.19 states that if (a, λ) satisfies the KKT conditions and a is a regular point, then the constrained optimization problem has a strict local maximum at $s = a$ provided that the Hessian of the Lagrange function, restricted to the binding constraints, is negative definite at that point. More

precisely, if we collect all m_1 functions that are binding at $s = a$ into the function g_1 , then for all $v \neq 0$ satisfying $Dg_1(a)v = 0$, we have $d(v) = v^T H \tilde{L} v < 0$. A proof of this theorem is requested in Exercise 4.16.

Theorem 4.15 gives rise to the next algorithm to solve our constrained maximization problem (4.8).

ALGORITHM 4.20 To find the candidate solutions of our constrained maximization problem (4.8) proceed as follows:

Step 1. Verify whether $f, g \in C^1$.

Step 2. Find all points, a , in the constraint set for which matrix G defined in (4.9) does not have full row rank.

Step 3. Write down the Lagrange function

$$L(s, \lambda_1, \dots, \lambda_m) := f(s) - \lambda_1(g_1(s) - b_1) - \dots - \lambda_m(g_m(s) - b_m).$$

Step 4. Determine all solutions $(a, \lambda_1, \dots, \lambda_m)$ that satisfy conditions 1, 2 and 3 from Theorem 4.15. All points identified in Steps 2 and 4 above are candidate solutions for the constrained optimization problem (4.8). To determine where a local or global maximum occurs, one can use, for example, Weierstrass' Theorem, Proposition 4.18 or Theorem 4.19. If these results do not provide a conclusive answer, the problem can be reconsidered with the additional constraint $\|s\| \leq r$, and it can then be verified whether the resulting global maximum remains unchanged for sufficiently large values of r .

EXAMPLE 4.21 Consider the maximization of $f(x, y) = -(x-2)^2 - (y-4)^2$, subject to the constraints

$$g_1(x, y) := x + y \leq 4, \quad g_2(x, y) := 2x + y \leq 6, \quad \tilde{g}_3(x, y) := x \geq 0 \text{ and } \tilde{g}_4(x, y) := y \geq 0.$$

In Figure 4.7, we plotted the feasible region.

To solve the problem, first, rewrite the problem in its standard form, by introducing $g_3(x, y) := -x \leq 0$ and $g_4(x, y) := -y \leq 0$.

Next, note that f is continuous and that the constraint set is compact. So, by Weierstrass' theorem, the problem has both a global maximum and a global minimum.

To find this maximum, we proceed along the lines of Algorithm 4.20.

Step 1. $Df = [-2(x-2) \quad -2(y-4)]$ and $Dg_1 = [1 \ 1]$; $Dg_2 = [2 \ 1]$; $Dg_3 = [-1 \ 0]$ and $Dg_4 = [0 \ -1]$.

Clearly, all partial derivatives are continuous. So, $f, g \in C^1$.

Step 2. Matrix G is in this case given by

$$G = \begin{bmatrix} 1 & 1 & x+y-4 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2x+y-6 & 0 & 0 \\ -1 & 0 & 0 & 0 & -x & 0 \\ 0 & -1 & 0 & 0 & 0 & -y \end{bmatrix}.$$

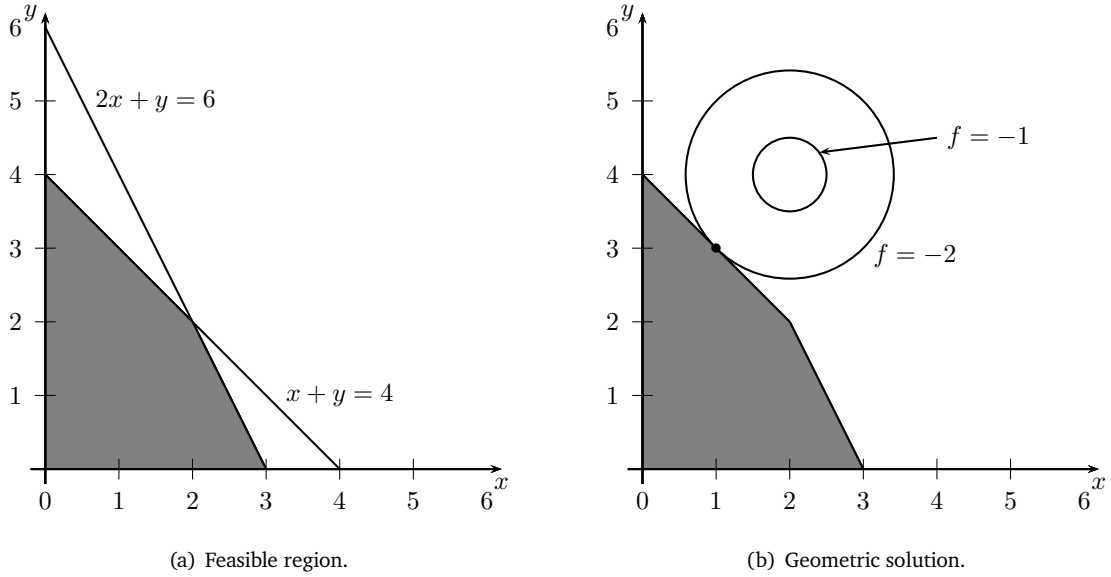


FIGURE 4.7 – Illustration Example 4.21.

By construction, for all points in the interior of the constraint set matrix G has full row rank. So, candidate solutions only occur at the boundary of the constraint set. We will inspect this boundary in a systematic way for candidate solutions. First, consider the case that precisely one constraint is binding. Note that $Dg_i(x, y) \neq [0 \ 0]$, $i = 1, \dots, 4$. So, the implicit function theorem always applies in that case. Consequently, without any further verification, we know matrix G has full row rank in that case.

Next, consider the case of two binding constraints. This yields the next intersection points (see also Figure 6, panel a): $(0, 0)$, $(3, 0)$, $(2, 2)$ and $(0, 4)$. By direct inspection one easily verifies that matrix G has full row rank in all these four cases too. So, step 2 does not provide any candidate solutions.

Another way to identify points at which matrix G has rank less than 4 is by applying Gaussian elimination. Using this method, one can easily verify that G has rank 4 if and only if the matrix

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -x & 0 \\ 0 & -1 & 0 & 0 & 0 & -y \\ 0 & 0 & x+y-4 & 0 & -x & -y \\ 0 & 0 & 0 & 2x+y-6 & -2x & -y \end{bmatrix}.$$

has rank 4. This is the case if and only if matrix

$$\begin{bmatrix} x+y-4 & 0 & -x & -y \\ 0 & 2x+y-6 & -2x & -y \end{bmatrix}.$$

has rank 2.

Clearly, if both $x + y - 4$ and $2x + y - 6$ are nonzero, this matrix has rank 2. Next, consider the case $x + y = 4$. In this situation, the matrix above has rank 2 if and only if the matrix

$$\begin{bmatrix} x-2 & -2x & x-4 \\ 0 & -x & x-4 \end{bmatrix}.$$

has rank 2. By further considering the cases $x = 2$ and $x \neq 2$, one easily verifies that this matrix always has rank 2.

Thus, this approach also confirms that the matrix G has rank 4 at all points in the constraint set.

Step 3. The Lagrange function is:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = -(x-2)^2 - (y-4)^2 - \lambda_1(x+y-4) - \lambda_2(2x+y-6) + \lambda_3x + \lambda_4y.$$

Step 4. The necessary conditions are:

1. $[-2(x-2) - \lambda_1 - 2\lambda_2 + \lambda_3 \quad -2(y-4) - \lambda_1 - \lambda_2 + \lambda_4] = [0 \ 0]$.
2. i) $\lambda_1(x+y-4) = 0$; ii) $\lambda_2(2x+y-6) = 0$; iii) $\lambda_3x = 0$; iv) $\lambda_4y = 0$.
3. i) $x + y \leq 4$; ii) $2x + y \leq 6$; iii) $x \geq 0$; iv) $y \geq 0$ and v) $\lambda_i \geq 0, i = 1, \dots, 4$.

To solve this set of (in)equalities, we proceed in a systematic way. First, we consider the case that none of the constraints is binding, second, we consider the case that precisely one constraint is binding, and, finally we consider the case that two constraints are simultaneously binding. Notice that it is not possible that more than two constraints are simultaneously binding.

Case 1: None of the constraints is binding. Or, $\lambda_i = 0, i = 1, \dots, 4$. Geometrically, in this case, we look for an optimum in the interior of the feasible set. That is, whether $f(x, y)$ has an unconstrained maximum on the interior of the feasible set.

From condition 1 it follows that $(x, y) = (2, 4)$, but this violates condition 3.i. So, this case does not provide a candidate solution.

Case 2: Precisely one constraint is binding. Geometrically, this means that we are looking for a solution on the boundary of the feasible set, excluding the corner points (where two constraints are binding).

Case 2.i: $g_1 = x + y = 4$ and the rest is not binding, so, by condition 3.v, $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

The set of necessary conditions reduces then to:

1. $[-2(x-2) - \lambda_1 \quad -2(y-4) - \lambda_1] = [0 \ 0]$.
3. i) $x + y = 4$; ii) $2x + y < 6$; iii) $x > 0$; iv) $y > 0$ and v) $\lambda_1 \geq 0$.

From 1, $\lambda_1 = -2(x-2) = -2(y-4)$. So, $x = y - 2$. Using 3.i) we conclude that $(x, y) = (1, 3)$ and, therefore, $\lambda_1 = 2$. This point also satisfies all other inequalities in condition 3, and is therefore a candidate solution.

Case 2.ii: $g_2 = 2x + y = 6$, while the remaining constraints are not. Hence, by condition 3.v, $\lambda_1 = \lambda_3 = \lambda_4 = 0$.

The set of necessary conditions reduces then to:

1. $[-2(x-2) - 2\lambda_2 \quad -2(y-4) - \lambda_2] = [0 \ 0]$.
3. i) $x + y < 4$; ii) $2x + y = 6$; iii) $x > 0$; iv) $y > 0$ and v) $\lambda_2 \geq 0$.

From 1, $\lambda_2 = -(x-2) = -2(y-4)$. So, $x = 2y-6$. Using 3.ii) we conclude that $(x, y) = (\frac{6}{5}, \frac{18}{5})$ and, therefore, $\lambda_2 = \frac{4}{5}$. However, this point violates condition 3.i). Therefore, this case does not provide a candidate solution.

Case 2.iii: $g_3 = x = 0$ and the rest is not binding, so, by condition 3.v, $\lambda_1 = \lambda_2 = \lambda_4 = 0$.

The set of necessary conditions reduces then to:

1. $[4 + \lambda_3 \quad -2(y-4)] = [0 \ 0]$.
3. i) $x + y < 4$; ii) $2x + y < 6$; iii) $x = 0$; iv) $y > 0$ and v) $\lambda_3 \geq 0$.

From 1, $\lambda_3 = -4$, which violates condition 3.v. So, this case does not provide a candidate solution.

Case 2.iv: $g_4 = y = 0$ and the rest is not binding, so, by condition 3.v, $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

The set of necessary conditions reduces then to:

1. $[-2(x-2) \quad 8 + \lambda_4] = [0 \ 0]$.
3. i) $x + y < 4$; ii) $2x + y < 6$; iii) $x > 0$; iv) $y = 0$ and v) $\lambda_4 \geq 0$.

From 1, $\lambda_4 = -8$, which violates condition 3.v. So, this case does not provide a candidate solution.

Case 3: Two of the constraints are simultaneously binding.

This occurs at one of the corner points $(0, 0)$, $(3, 0)$, $(2, 2)$ and $(0, 4)$. It is easily verified that, at all four points, the set of necessary conditions has no solution. The constraint $\lambda_i \geq 0$, $i = 1, \dots, 4$, is violated in all cases.

Thus, the only candidate solution is $(1, 3)$. As noted earlier, by Weierstrass' theorem the problem admits a solution. Consequently, f attains its global maximum at $(1, 3)$. Figure 4.7, panel b, illustrates the solution geometrically.

EXAMPLE 4.22 Consider the maximization of $f(x, y) = x^2y$, subject to the constraints

$$g_1(x, y) := x^2 + y^2 \leq 25 \text{ and } \tilde{g}_2(x, y) := y \geq \frac{4}{3}x.$$

In Figure 4.8, we plotted in gray the feasible region.

Clearly, the constraint set is compact. Since, moreover, f is continuous, by Weierstrass' theorem, the problem has both a global maximum and a global minimum.

To find this maximum, we proceed along the lines of Algorithm 4.20. To that end, we first rewrite the problem into the standard format. That is, introduce $g_1(x, y) := x^2 + y^2 \leq 25$ and $g_2(x, y) := \frac{4}{3}x - y \leq 0$. With this, the problem fits into the framework of (4.8).

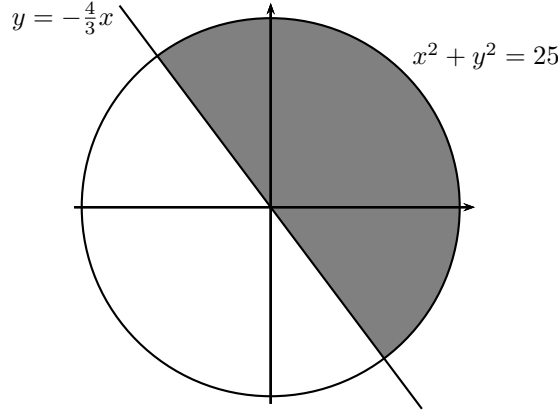


FIGURE 4.8 – Feasible set Example 4.22.

Step 1. $Df = [2xy \ x^2]$ and $Dg_1 = [2x \ 2y]$; $Dg_2 = [-\frac{4}{3}x \ -1]$. Clearly, all partial derivatives are continuous. So, $f, g \in C^1$.

Step 2. Matrix G equals in this case

$$G = \begin{bmatrix} 2x & 2y & x^2 + y^2 - 25 & 0 \\ \frac{4}{3} & -1 & 0 & \frac{4}{3}x - y \end{bmatrix}.$$

As we already observed in the previous Example 4.21, the search for candidate solutions can be restricted to the boundary of the constraint set. Therefore, first, consider the case of one binding constraint. Note that $Dg_1 = [0 \ 0]$ if $(x, y) = (0, 0)$. But, it is easily verified that, if $(x, y) = (0, 0)$ matrix G has rank two. Since $Dg_2 \neq [0 \ 0]$ we conclude that, if one constraint is binding, matrix G will always be full row rank.

In case both constraints are binding, the intersection points are obtained by, e.g., substitution of $y = -\frac{4}{3}x$ into the equation $x^2 + y^2 = 25$. This yields the intersection points $(-3, 4)$ and $(3, -4)$. A direct inspection of matrix G shows that in those two points its rank is two. So, we don't get any candidate solutions in this step.

Also in this case we illustrate how Gaussian elimination can be used to reach the same conclusion. To this end, note that the matrix G has rank 2 if and only if the matrix

$$\begin{bmatrix} 1 & -\frac{3}{4} & 0 & x - \frac{3}{4}y \\ 2x & 2y & x^2 + y^2 - 25 & 0 \end{bmatrix}.$$

has rank 2. By subtracting $2x$ times the first row from the second one, we find that this matrix has rank 2 if and only if matrix

$$\begin{bmatrix} 1 & -\frac{3}{4} & 0 & x - \frac{3}{4}y \\ 0 & \frac{3}{2}x + 2y & x^2 + y^2 - 25 & -2x(x - \frac{3}{4}y) \end{bmatrix}$$

has rank 2.

Clearly, this matrix has rank less than 2 if and only if

$$\frac{3}{2}x + 2y = 0, \quad x^2 + y^2 - 25 = 0, \quad \text{and} \quad -2x(x - \frac{3}{4}y) = 0$$

hold simultaneously. It is straightforward to verify that this never occurs at any point in the constraint set.

Step 3. The Lagrange function is:

$$L(x, y, \lambda_1, \lambda_2) = x^2y - \lambda_1(x^2 + y^2 - 25) + \lambda_2(\frac{4}{3}x + y).$$

Step 4. The necessary conditions are:

1. $[2xy - 2\lambda_1x + \frac{4}{3}\lambda_2 \quad x^2 - 2\lambda_1y + \lambda_2] = [0 \quad 0]$.
2. i) $\lambda_1(x^2 + y^2 - 25) = 0$; ii) $\lambda_2(\frac{4}{3}x + y) = 0$.
3. i) $x^2 + y^2 \leq 25$; ii) $y \geq \frac{4}{3}x$; iii) $\lambda_i \geq 0, i = 1, 2$.

To solve this system of (in)equalities, we proceed in a systematic manner. First, we consider the case in which none of the constraints is binding. Second, we examine the case in which exactly one constraint is binding. Finally, we analyze the case in which two constraints are binding simultaneously.

Case 1: None of the constraints is binding. Or, $\lambda_i = 0, i = 1, 2$. So, geometrically, we first consider the case again whether $f(x, y)$ has an unconstrained maximum on the interior of the feasible set. The set of necessary conditions reduce then to:

1. $[2xy \quad x^2] = [0 \quad 0]$.
2. i) $x^2 + y^2 < 25$; ii) $y \geq \frac{4}{3}x$; iii) $\lambda_i = 0, i = 1, 2$.

From condition 1 we get $x = 0$, whereas, by condition 3.i $0 < y < 5$. This yields a half line of candidate solutions. The value f attains on this half line is zero.

Case 2: Precisely one constraint is binding. Geometrically, this means that we look for a solution on the boundary of the feasible set, excluding the corner points.

Case 2.i: $g_1 = x^2 + y^2 = 25$ and g_2 is not binding, so, by condition 3.iii, $\lambda_2 = 0$.

The set of necessary conditions reduces then to:

1. $[2xy - 2\lambda_1x \quad x^2 - 2\lambda_1y] = [0 \quad 0]$.
2. i) $x^2 + y^2 = 25$; ii) $y > \frac{4}{3}x$; iii) $\lambda_1 \geq 0$.

From condition 1, first equation, we have $2x(y - \lambda_1) = 0$. So, either $x = 0$ or $y = \lambda_1$.

Case 2.i.I: $x = 0$.

If $x = 0$, by condition 1, second equation, either $\lambda_1 = 0$ or $y = 0$. Note, $(0, 0)$ violates condition 3.i. If, $\lambda_1 = 0$, since $x = 0$, it follows from condition 3.i and 3.ii that $y = 5$. So, a candidate solution is $(0, 5)$ with $\lambda_1 = \lambda_2 = 0$. Note, $f(0, 5) = 0$.

Case 2.i.II: $y = \lambda_1$.

From condition 1, second equation, it follows then that $x^2 = 2\lambda_1^2$. So, $x = \sqrt{2}\lambda_1$ or $x = -\sqrt{2}\lambda_1$. Furthermore, by condition 3.i, we have $3\lambda_1^2 = 25$. By condition 3.iii, it follows

then that $\lambda_1 = \frac{5}{\sqrt{3}}$. Since $(\sqrt{2}\lambda_1, \lambda_1)$ does not satisfy condition 3.ii), we have additionally one candidate solution: $(-5\sqrt{\frac{2}{3}}, \frac{5}{\sqrt{3}})$. Note $f(-5\sqrt{\frac{2}{3}}, \frac{5}{\sqrt{3}}) = \frac{250}{3\sqrt{3}}$.

Case 2.ii: $g_2 = \frac{4}{3}x + y = 0$ and g_1 is not binding, so, by condition 3.iii, $\lambda_1 = 0$.

The set of necessary conditions reduces then to:

1. $[2xy + \frac{4}{3}\lambda_2 \quad x^2 + \lambda_2] = [0 \quad 0]$.
3. i) $x^2 + y^2 < 25$; ii) $\frac{4}{3}x + y = 0$; iii) $\lambda_2 \geq 0$.

By condition 1, the second equation, and condition 3.iii it follows that $x = \lambda_2 = 0$. Then, by condition 3.ii, $y = 0$. Hence, $(0, 0)$ is a candidate solution. Note that $f(0, 0) = 0$.

Case 3: Two of the constraints are binding simultaneously.

This occurs at one of the corner points: $(-3, 4)$ or $(3, -4)$. It is easily verified that, at both points, the set of necessary conditions has no solution since the constraint $\lambda_i \geq 0$, $i = 1, 2$ is violated in both cases.

Finally, by comparing the value at the different candidate solutions, we conclude that the maximum is attained at $(-5\sqrt{\frac{2}{3}}, \frac{5}{\sqrt{3}})$.

The next example illustrates a situation where the constrained optimization problem does not have a global maximum, but does have a local maximum.

EXAMPLE 4.23 Consider the maximization of $f(x, y) = x^2y - 2y$, subject to the constraints

$$\tilde{g}_1(x, y) := y \geq x^2 \text{ and } \tilde{g}_2(x, y) := x \geq -2.$$

First, note that the feasible set is not compact. Moreover, by choosing x arbitrarily large and setting $y = x^2$, the objective function

$$f(x, y) = x^4 - 2x^2$$

becomes arbitrarily large. Hence, the problem does not admit a global maximum.

With $g_1 := x^2 - y$ and $g_2 := -x - 2$ the problem fits into the standard framework (4.8). Following the algorithm we have:

Step 1.

$$Df = [2xy \quad x^2 - 2] \text{ and } Dg = \begin{bmatrix} 2x & -1 \\ -1 & 0 \end{bmatrix}.$$

All partial derivatives are continuous; therefore $f, g \in C^1$.

Step 2. Check whether the matrix

$$G := \begin{bmatrix} 2x & -1 & x^2 - y & 0 \\ -1 & 0 & 0 & -x - 2 \end{bmatrix}$$

has rank 2.

By construction, G has full row rank at all interior points, so it suffices to check boundary points.

First, consider the case in which only one constraint is binding. It is easily verified that G has rank 2 in both cases.

If both constraints are binding, then $(x, y) = (-2, 4)$. Again, one readily verifies that G has rank 2 at this point.

So we don't find any candidate locations in this step.

Notice that, using Gaussian elimination to verify the rank of the matrix G , we can inspect the rank of matrix

$$\begin{bmatrix} -1 & 0 & 0 & -x - 2 \\ 2x & -1 & x^2 - y & 0 \end{bmatrix}.$$

Equivalently, we can consider the rank of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & x + 2 \\ 0 & -1 & x^2 - y & -2x(x + 2) \end{bmatrix}.$$

which clearly always has rank 2. Hence, this approach also confirms that there are no additional candidate points to consider.

Step 3.

$$\mathcal{L} = x^2y - 2y - \lambda_1(x^2 - y) - \lambda_2(-x - 2).$$

Step 4. From $\mathcal{L}$ we obtain next Kuhn-Tucker conditions:

$$\begin{cases} [2xy - 2\lambda_1x + \lambda_2, x^2 - 2 + \lambda_1] = [0 \ 0]; \\ \lambda_1(x^2 - y) = 0; \\ \lambda_2(-x - 2) = 0; \\ \lambda_i \geq 0 \ i = 1, 2; \\ y \geq x^2 \text{ and } x \geq -2. \end{cases}$$

If the first constraint is not binding, then $\lambda_1 = 0$. In this case the first KKT condition has no solution.

If $y = x^2$ but the second constraint is not binding, then $\lambda_2 = 0$. Substituting these values into the first condition shows then the next solutions satisfy all KKT conditions: $(x, y, \lambda_1, \lambda_2) = (1, 1, 1, 0)$, $(x, y, \lambda_1, \lambda_2) = (-1, 1, 1, 0)$ and $(x, y, \lambda_1, \lambda_2) = (0, 0, 2, 0)$.

Finally, if both constraints are binding, we obtain $(x, y) = (-2, 4)$. However, this point violates the first KKT condition.

Hence, we get three candidate solutions for a local maximum. That is: $(x, y, \lambda_1, \lambda_2) = (1, 1, 1, 0)$, $(x, y, \lambda_1, \lambda_2) = (-1, 1, 1, 0)$ and $(x, y, \lambda_1, \lambda_2) = (0, 0, 2, 0)$.

To determine whether the problem has a local maximum at any of these points, we apply Theorem 4.19. Note that at all three points, g_1 is binding, while g_2 is not.

First, consider $(x, y, \lambda_1, \lambda_2) = (1, 1, 1, 0)$. Following Theorem 4.19, we define

$$H\tilde{L} := Hf - Hg_1 = \begin{bmatrix} 2y & 2x \\ 2x & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B := [1 \ 2x]^T$$

evaluated at $(x, y) = (1, 1)$. Then,

$$B^T(1, 1)H\tilde{L}(1, 1)B(1, 1) = 8 > 0.$$

Hence, the constrained optimization problem does **not** have a local maximum at $(1, 1)$.

Similarly, for $(x, y, \lambda_1, \lambda_2) = (-1, 1, 1, 0)$, we find

$$B^T(-1, 1)H\tilde{L}(-1, 1)B(-1, 1) = 8 > 0.$$

so there is also **no** local maximum at $(-1, 1)$.

Finally, at $(x, y, \lambda_1, \lambda_2) = (0, 0, 2, 0)$, we have

$$H\tilde{L} := Hf - 2Hg_1 = \begin{bmatrix} 2y & 2x \\ 2x & 0 \end{bmatrix} - 2 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B := [1 \ 2x]^T$$

evaluated at $(x, y) = (0, 0)$. Here,

$$B^T(0, 0)H\tilde{L}(0, 0)B(0, 0) = -4 < 0,$$

so the constrained optimization problem does have a local maximum at $(0, 0)$.

EXAMPLE 4.24 Consider a consumer who wants to maximize the utility function

$$U(x, y) = cx^\alpha y^{1-\alpha}$$

subject to the budget constraint

$$p_1x + p_2y \leq I,$$

where, I is the consumer's income, p_1 is the price of good x , and p_2 is the price of good y . Additionally, the consumer must purchase at least one unit of each good (which implicitly implies that we assume that $p_1 + p_2 < I$).

We can reformulate this problem in the standard constrained optimization framework by defining the constraints:

$$g_1(x, y) := p_1x + p_2y \leq I, \quad g_2(x, y) := -x \leq -1, \quad \text{and} \quad g_3(x, y) := -y \leq -1.$$

In Figure 4.9, we plotted the feasible region.

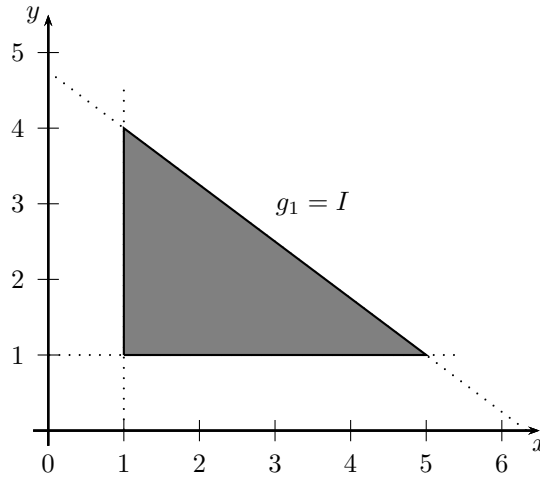


FIGURE 4.9 – Feasible region in Example 4.24.

Since $U(x, y)$ is continuous and the constraint set is compact, Weierstrass' theorem ensures that the problem attains both a global maximum and a global minimum.

To find this maximum, we proceed along the lines of Algorithm 4.20.

Step 1. $DU = [c\alpha x^{\alpha-1}y^{1-\alpha} \quad cx^\alpha y^{-\alpha}]$ and $Dg_1 = [p_1 \ p_2]$; $Dg_2 = [-1 \ 0]$ and $Dg_3 = [0 \ -1]$. Clearly, all partial derivatives are continuous on the feasible region. So, $f, g \in C^1$.

Step 2. Matrix G equals in this case

$$G = \begin{bmatrix} p_1 & p_2 & p_1x + p_2y - I & 0 & 0 \\ -1 & 0 & 0 & 1 - x & 0 \\ 0 & -1 & 0 & 0 & 1 - y \end{bmatrix}.$$

As argued in Example 4.21, to find candidate solutions, it suffices to inspect the rank of matrix G on the boundaries of the constraint set.

First, consider the case where only one constraint is binding. Note that $Dg_i(x, y) \neq [0 \ 0]$, for $i = 1, \dots, 3$. As argued in Example 4.21, the matrix G is then full row rank, so no candidate solutions arise in this case.

Next, consider the case where two constraints are binding. The intersection points are $(1, 1)$, $(1, \frac{I-p_1}{p_2})$, and $(\frac{I-p_2}{p_1}, 1)$. A direct inspection of matrix G at these points shows that it is full row rank at all three locations, so none of these points qualify as candidate solutions either.

Thus, no additional candidate solutions are obtained in this step.

Alternatively, one can also verify the rank of matrix G using Gaussian elimination. It can be easily shown that G has rank 3 if and only if the following matrix has rank 3

$$\begin{bmatrix} 1 & 0 & 0 & x - 1 & 0 \\ 0 & 1 & 0 & 0 & y - 1 \\ 0 & 0 & p_1x + p_2y - I & p_1(1 - x) & p_2(1 - y) \end{bmatrix}.$$

This matrix fails to have full row rank if and only if all of the following hold simultaneously

$$p_1x + p_2y - I = 0, \quad p_1(1 - x) = 0, \quad \text{and} \quad p_2(1 - y) = 0.$$

It is easy to verify that this is impossible, which confirms our previous observation that no additional candidate solutions are obtained in this step.

Step 3. The Lagrange function is:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = cx^\alpha y^{1-\alpha} - \lambda_1(p_1x + p_2y - I) + \lambda_2(x - 1) + \lambda_3(y - 1).$$

Step 4. The necessary conditions are:

1. $[c\alpha x^{\alpha-1}y^{1-\alpha} - \lambda_1p_1 + \lambda_2 \quad c(1-\alpha)x^\alpha y^{-\alpha} - \lambda_1p_2 + \lambda_3] = [0 \ 0]$.
2. i) $\lambda_1(p_1x + p_2y - I) = 0$; ii) $\lambda_2(x - 1) = 0$; iii) $\lambda_3(y - 1) = 0$.
3. i) $p_1x + p_2y \leq I$; ii) $x \geq 1$; iii) $y \geq 1$; iv) $\lambda_i \geq 0$, $i = 1, 2, 3$.

To solve this set of (in)equalities we consider the next cases.

Case 1: None of the constraints is binding, i.e., $p_1x + p_2y < I$; ii) $x > 1$; and iii) $y > 1$.

Then, $\lambda_i = 0$, $i = 1, 2, 3$ and the set of necessary conditions reduces to

1. $[c\alpha x^{\alpha-1}y^{1-\alpha} \quad c(1-\alpha)x^\alpha y^{-\alpha}] = [0 \ 0]$.
3. i) $p_1x + p_2y < I$; ii) $x > 1$; and iii) $y > 1$.

It is easily seen that this set of equations has no solution.

Case 2: Precisely one constraint is binding.

Case 2.i: $p_1x + p_2y = I$ and the other two constraints are not binding. So, by condition 3.iv), $\lambda_2 = \lambda_3 = 0$.

Then, the set of necessary conditions reduces to

$$1. [c\alpha x^{\alpha-1}y^{1-\alpha} - \lambda_1 p_1 \quad c(1-\alpha)x^\alpha y^{-\alpha} - \lambda_1 p_2] = [0 \ 0].$$

$$3. \text{ i) } p_1x + p_2y = I; \text{ ii) } x > 1; \text{ iii) } y > 1; \text{ and iv) } \lambda_1 \geq 0.$$

From item 1 we have $\lambda_1 p_1 = c\alpha x^{\alpha-1}y^{1-\alpha}$ and $\lambda_1 p_2 = c(1-\alpha)x^\alpha y^{-\alpha}$. Note, $\lambda_1 > 0$, for otherwise the set of equations has no solution. Dividing both equations yields:

$$\frac{p_1}{p_2} = \frac{c\alpha x^{\alpha-1}y^{1-\alpha}}{c(1-\alpha)x^\alpha y^{-\alpha}}, \text{ or, } \frac{p_1}{p_2} = \frac{\alpha y}{(1-\alpha)x}.$$

Substitution of $y = \frac{(1-\alpha)p_1}{\alpha p_2}x$ into the budget constraint 3.i shows that $x = \frac{\alpha}{p_1}I$ and, consequently, $y = \frac{1-\alpha}{p_2}I$. Furthermore,

$$\lambda_1 = \frac{c\alpha}{p_1} \left(\frac{y}{x}\right)^{1-\alpha} = \frac{c\alpha}{p_1} \left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{1-\alpha} = c \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}.$$

Therefore, this is an appropriate candidate solution.

Case 2.ii: $x = 1$ and the other two constraints are not binding. So, by condition 3.iv), $\lambda_1 = \lambda_3 = 0$.

Then, the set of necessary conditions reduces to

$$1. [c\alpha y^{1-\alpha} + \lambda_2 \quad c(1-\alpha)y^{-\alpha}] = [0 \ 0].$$

$$3. \text{ i) } p_1 + p_2y < I; \text{ ii) } x = 1; \text{ iii) } y > 1; \text{ iv) } \lambda_2 \geq 0.$$

From condition 1, it is clear that this case does not have a solution.

Case 2.iii: $y = 1$ and the other two constraints are not binding. So, by condition 3.iv), $\lambda_1 = \lambda_2 = 0$.

Similar to Case 2.ii, it follows then that there is no solution satisfying all conditions.

Case 3: Two of the constraints are simultaneously binding.

This happens either at one of the corner points: $(1, 1)$; $(1, \frac{I-p_1}{p_2})$ or $(\frac{I-p_2}{p_1}, 1)$. It is easy to verify that, at all three points, the set of necessary conditions has no solution. In each case, the constraint $\lambda_i \geq 0$, for $i = 1, 2, 3$, is violated.

Since we identify only a single candidate solution, it follows from Weierstrass' theorem that the maximum is given by

$$U\left(\frac{\alpha}{p_1}I, \frac{1-\alpha}{p_2}I\right) = c \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} I.$$

EXERCISE 4.3 Solve the next constrained optimization problems.

- Maximize $f(x, y) = x + y$, subject to the constraints $x^2 + 2y^2 \leq 6$, $x \geq 0$ and $y \geq 0$.
- Maximize $f(x, y) = -x^2 - xy - y^2$ subject to the constraints $g_1(x, y) = x - 2y \leq -1$; $g_2(x, y) = 2x + y \leq 2$; $x \geq 0$ and $y \geq 0$.
- Maximize $f(x, y) = y^2 - x^2$, subject to the constraints $g_1(x, y) = x^2 + y^2 \leq 4$ and $g_2(x, y) = x^2 + (y - 2)^2 \leq 4$.
- Minimize $f(x, y) = x + xy$, subject to the constraints $y \leq 2 - x^2$, $x \geq 0$ and $y \geq 0$.
- Maximize $f(x, y) = x + y$, subject to the constraint $g(x, y) = y \leq -\frac{1}{2}x^2$.
- Maximize $f(x, y) = x - \frac{1}{3}xy$, subject to the constraints $y \geq x^2$ and $x \leq 2$.

4.4 Some pathological examples

In this section, we present examples that highlight the importance of Step 2 in Algorithm 4.20. While the examples are stylized, they clearly demonstrate that correctly executing Step 2 can be critical for identifying the optimum of a problem.

The first example shows that a solution may occur at a "corner" point - i.e., a point where the number of active constraints equals the number of variables - even when this "corner" point does not satisfy the first-order Kuhn-Tucker (in)equalities.

EXAMPLE 4.25 Consider the maximization of $f(x, y) = x$ subject to the constraints

$$g_1(x, y) := x \geq 0, \quad g_2(x, y) := y \geq 0 \quad \text{and} \quad g_3(x, y) := y - (1 - x)^3 \leq 0.$$

In Figure 4.10, we plotted the feasible region.

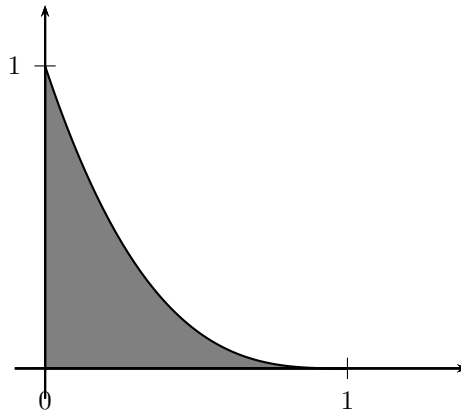


FIGURE 4.10 – Feasible region in Example 4.25.

From this figure it is clear that the constraint set is compact. Since, moreover, f is continuous we know that this problem has a solution. In fact, since $y \geq 0$, it follows immediately from $g_3 \leq 0$ that $(1-x)^3 \geq 0$. That is, $x \leq 1$. Therefore, $f(x, y)$ attains its global maximum at $(x, y) = (1, 0)$. We next demonstrate how this result follows using Algorithm 4.20.

Step 1. $Df = [1 \ 0]$ and $Dg_1 = [1 \ 0]$; $Dg_2 = [0 \ 1]$; and $Dg_3 = [3(1-x)^2 \ 1]$. Clearly, all partial derivatives are continuous. So, $f, g \in C^1$.

Step 2. Matrix G equals in this case

$$G = \begin{bmatrix} -1 & 0 & -x & 0 & 0 \\ 0 & -1 & 0 & -y & 0 \\ 3(1-x)^2 & 1 & 0 & 0 & y - (1-x)^3 \end{bmatrix}.$$

As noted in Example 4.21, to find candidate solutions, it suffices to inspect the rank of matrix G on the boundaries of the constraint set.

First, consider the case where only one constraint is binding. Note that $Dg_i(x, y) \neq [0 \ 0]$, $i = 1, 2, 3$. As discussed in Example 4.21, in these cases the matrix G will have full row rank.

Next, consider the case where two constraints are binding. This yields the intersection points (see also Figure 9): $(0, 0)$, $(1, 0)$ and $(0, 1)$. The rank of matrix G at $(1, 0)$ is two, while at $(0, 0)$ and $(0, 1)$ it is three. Hence, in this step we obtain one candidate solution: $(1, 0)$.

Step 3. The Lagrange function is:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = x + \lambda_1 x + \lambda_2 y - \lambda_3(y - (1-x)^3).$$

Step 4. The necessary conditions are:

1. $[1 + \lambda_1 - 3\lambda_3(1-x)^2 \ \lambda_2 - \lambda_3] = [0 \ 0]$.
2. i) $\lambda_1 x = 0$; ii) $\lambda_2 y = 0$; iii) $\lambda_3(y - (1-x)^3) = 0$.
3. i) $x \geq 0$; ii) $y \geq 0$; iii) $y - (1-x)^3 \leq 0$ and iv) $\lambda_i \geq 0$, $i = 1, \dots, 3$.

By item 1, $\lambda_2 = \lambda_3$. In case $\lambda_3 = 0$, it follows from item 1 that $\lambda_1 = -1 < 0$, which contradicts item 3.iv. So, $\lambda_2 = \lambda_3 > 0$ and, consequently by item 2, $y = 0$ and $y = (1-x)^3$. From which we conclude that necessarily $x = 1$. However, this again implies, by item 1, that $\lambda_1 < 0$, which is not permitted. Therefore, we conclude that Step 4 does not provide any additional candidate solutions.

So, the only candidate solution is the one we found in Step 2, i.e.: $(1, 0)$. Therefore, by Weierstrass' theorem, the maximum is attained at $(1, 0)$.

In the next exercise, the reader is asked to demonstrate that a solution can occur at a point where two constraints are active, even though this point does not satisfy the first-order Kuhn-Tucker (in)equalities. Moreover, this point is neither a solution with only one active constraint nor a "corner" point.

EXERCISE 4.4 Solve the optimization problem: maximize $f(x, y, z) = x - (z-1)^2$, w.r.t. x, y, z , subject to the constraints $y - (1-x)^3 \leq 0$, $x \geq 0$, $y \geq 0$ and $0 \leq z \leq 2$.

The following example illustrates a case in which the optimization problem achieves its maximum at a point for which the Implicit Function Theorem is not applicable and the first-order Kuhn–Tucker (in)equalities fail to hold.

EXAMPLE 4.26 Consider the maximization of $f(x, y) = x$ subject to the constraints

$$g_1(x, y) := y^2 + (x-1)^3 \leq 0, \text{ and } g_2(x, y) := x \geq 0.$$

In Figure 4.11, we plotted the feasible region.

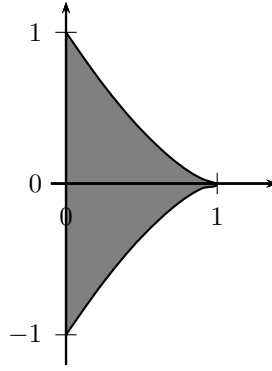


FIGURE 4.11 – Feasible region in Example 4.26.

From this figure it is clear that the constraint set is compact. Since, moreover, f is continuous we know that this problem has a solution. Since $y^2 \geq 0$ it follows directly from $g_1 \leq 0$ that $(x-1)^3 \leq 0$. That is, $x \leq 1$. Therefore, $f(x, y)$ attains its global maximum at $(x, y) = (1, 0)$. We next demonstrate how this result follows using Algorithm 4.20.

Step 1. $Df = [1 \ 0]$, $Dg_1 = [2y \ 3(x-1)^2]$ and $Dg_2 = [1 \ 0]$. Clearly, all partial derivatives are continuous.

So, $f, g \in C^1$.

Step 2. Matrix G equals in this case

$$G = \begin{bmatrix} 3(x-1)^2 & 2y & y^2 + (x-1)^3 & 0 \\ -1 & 0 & 0 & -x \end{bmatrix}.$$

Following the argumentation from Example 4.21 again we notice that, to find candidate solutions, it suffices to inspect the rank of matrix G on the boundaries of the constraint set.

First, consider the case of one binding constraint. Note that $Dg_2(x, y) \neq [0 \ 0]$. On the other hand, $Dg_1(x, y) = [0 \ 0]$ at $(x, y) = (1, 0)$, and G has rank 1 at $(1, 0)$. So, $(1, 0)$ is a candidate optimal solution.

Next, consider the case of two binding constraints. This yields the intersection points (see also Figure 10): $(0, -1)$ and $(0, 1)$. Again, by a direct inspection, one verifies that the rank of G is two in both points. So, these points don't apply as candidate solutions.

Step 3. The Lagrange function is:

$$L(x, y, \lambda_1, \lambda_2) = x - \lambda_1(y^2 + (x - 1)^3) + \lambda_2 x.$$

Step 4. The necessary conditions are:

1. $[1 - 3\lambda_1(x - 1)^2 + \lambda_2 \quad -2\lambda_1 y] = [0 \quad 0]$.
2. i) $\lambda_1(y^2 + (x - 1)^3) = 0$; ii) $\lambda_2 x = 0$.
3. i) $y^2 + (x - 1)^3 \leq 0$; ii) $x \geq 0$; and iii) $\lambda_i \geq 0$, $i = 1, 2$.

From item 1, first entry, it follows that $\lambda_1 > 0$. Therefore, by item 2.i, $y^2 + (x - 1)^3 = 0$ and, by item 1 (second entry), $y = 0$. But this implies that $x = 1$. This violates item 1, first entry. Therefore, we conclude that Step 4 does not provide any additional candidate solutions.

So, the only candidate solution (and thus, by Weierstrass' theorem, the maximum) is the one we found in Step 2, i.e.: $(1, 0)$.

Exercise 4.5 below illustrates a case in which the solution of the optimization problem is attained at a point where only a single constraint is active. At this point, the first-order Kuhn–Tucker (in)equalities are not satisfied. Moreover, there exists a connected set of points—containing the point at which the optimum occurs—at which the conditions for applying the Implicit Function Theorem are not satisfied.

EXERCISE 4.5 Solve the optimization problem: maximize $f(x, y, z) = x - (z - 1)^2$, w.r.t. x, y, z , subject to the constraints $y^2 + (x - 1)^3 \leq 0$, $x \geq 0$ and $0 \leq z \leq 2$.

4.5 Mixed constraints: a shortcut

As noted in the introduction to Section 3, the theory developed there can also be applied to constrained optimization problems involving both equality and inequality constraints. As previously observed, such problems can, in principle, be handled by rewriting each equality constraint as two inequality constraints. However, this approach typically introduces additional variables and thus complicates the analysis. This complication can be avoided by combining the theory developed in Chapters 3 and 4. The resulting approach is the subject of the present section.

Assume the set S is determined by the set of (in)equalities

$$S := \{x \in \mathbb{R}^n \mid g_i(x) \leq b_i, \ i = 1, \dots, k, \text{ and } h_j(x) = c_j, \ j = 1, \dots, m\}. \quad (4.10)$$

Let the constraint set D be given by $D := C \cap S$. Consider the problem

Problem Statement Let $f, g_i, i = 1, \dots, k$, and $h_j, j = 1, \dots, m$, be continuous differentiable functions on an open set $C \subset \mathbb{R}^n$.

$$\text{Maximize } f(s), \text{ subject to } s \in D. \quad (4.11)$$

Building on the preceding results, we obtain the following corollary to Theorem 4.15.

THEOREM 4.27 Consider the constrained maximization problem (4.11). Assume the problem has a solution at $s = a$. Let $L(s, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m) :=$

$$f(s) - \lambda_1(g_1(s) - b_1) - \dots - \lambda_k(g_k(s) - b_k) - \mu_1(h_1(s) - c_1) - \dots - \mu_m(h_m(s) - c_m) \quad (4.12)$$

be the with this problem associated Lagrange function. Then, if at $s = a$ the derivatives of the set of constraints that are binding are linearly independent¹² (i.e. a is regular), there exist $\lambda_i, \mu_i \in \mathbb{R}$ such that $(a, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m)$ satisfy the next set of (in)equalities (KKT)

1. $[D_1 L(a, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m) \dots D_n L(a, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m)] = [0 \dots 0]$.
 2. $\lambda_i(g_i(a) - b_i) = 0$.
 3. $g_i(a) \leq b_i$ and $\lambda_i \geq 0$.
 4. $h_i(a) = c_i$.
- (KKT)

REMARK 4.28 The condition in Theorem 4.27 requiring that, at the maximum $s = a$, the matrix of derivatives of the active constraints has full row rank can be replaced by an alternative full row rank condition, which is simpler to state. Specifically, this condition can be reformulated as the requirement that the matrix $\tilde{G}$, defined below, has full row rank at $s = a$.

$$\tilde{G} = \begin{bmatrix} Dg(s) & \text{diag}(g_i(s) - b_i) \\ Dh(s) & 0 \end{bmatrix}. \quad (4.13)$$

Following the approach of the proof of Proposition 4.11, we obtain the sufficient condition for the existence of a global maximum stated in Proposition 4.29 below. Furthermore, by applying the reasoning from Chapter 3 and Sections 4.2 and 4.3, we derive sufficient conditions for the existence of a local maximum for the mixed-constraint optimization problem, as presented in Theorem 4.30.

¹² Note this is equivalent with the statement that the matrix consisting of the derivatives of the set of constraints that are binding has full row rank

PROPOSITION 4.29 *Let f , g and h be differentiable on an open convex set C . Assume (a, λ_0, μ_0) are a solution of the set of Kuhn-Tucker (KKT) conditions. Then, if $\tilde{L}(x) := L(x, \lambda_0, \mu_0)$ is concave on C , the constrained maximization problem (4.8) has a global maximum at $s = a$.*

THEOREM 4.30 *Consider the mixed constrained maximization problem (4.11) and its associated Lagrange function (4.12). Let $(a, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m)$ satisfy the Kuhn-Tucker (in)equalities (KKT). Assume, without loss of generality, that the first k_1 constraints g_i , $i = 1, \dots, k_1$, are binding at $x = a$, while the remaining constraints are non-binding. Define*

$$\tilde{g}_1(x) := [g_1 \cdots g_{k_1} \ h_1 \cdots h_m]^T.$$

Suppose that $D\tilde{g}_1(a)$ has full row rank $k_1 + m$. Write

$$\tilde{g}'_1(a) =: [D_x \tilde{g}_1(a) \ D_y \tilde{g}_1(a)],$$

where $D_y \tilde{g}_1(a)$ denotes the final $(k_1 + m) \times (k_1 + m)$ block of the matrix $\tilde{g}'_1(a)$. Assume, again without loss of generality, that $D_y \tilde{g}_1(a)$ is invertible.

Define

$$H\tilde{L} := Hf(a) - \sum_{i=1}^k \lambda_i Hg_i(a) - \sum_{i=1}^m \mu_i Hh_i(a) \text{ and } B := \begin{bmatrix} I_{(n-(k_1+m)) \times (n-(k_1+m))} \\ -(D_y \tilde{g}_1(a))^{-1} * D_x \tilde{g}_1(a) \end{bmatrix},$$

where $I_{(n-(k_1+m)) \times (n-(k_1+m))}$ denotes the identity matrix in $\mathbb{R}^{(n-(k_1+m)) \times (n-(k_1+m))}$.

Then $f(a)$ is a strict local maximum of the constrained optimization problem if $B^T * H\tilde{L} * B$ is negative definite. If $B^T * H\tilde{L} * B$ is not negative semi-definite, the constrained optimization problem does not admit a local maximum at $x = a$.

As in Theorem 4.19, Theorem 4.30 admits a reformulation based on the nullspace of the derivative matrix of the active constraints.

If (a, λ, μ) is a solution of the KKT conditions and a is a regular point with respect to the active constraints, then the constrained optimization problem has a strict local maximum at $s = a$ provided that the Hessian of the Lagrange function, restricted to the nullspace of the active constraints, is negative definite. More precisely, if we collect all $k_1 + m$ functions that are active at $s = a$ into the function g_1 , then for every nonzero vector v satisfying $Dg_1(a)v = 0$, we have $d(v) = v^T H\tilde{L}v < 0$. Conversely, if there exists a vector v in this set such that $d(v) > 0$, then a cannot be a maximum location.

To solve the constrained maximization problem (4.11) one can proceed along the lines of Algorithm 4.20, with matrix G replaced by matrix $\tilde{G}$. We illustrate this in the next example.

EXAMPLE 4.31 Consider the maximization of $f(x, y) = x + 2y$ over the set

$$\Gamma = \{(x, y) \mid g_1(x, y) := x^2 + y^2 - 2 \leq 0, g_2(x, y) := -x \leq 0, h(x, y) := x - y = 0\}.$$

Observe that Γ describes the line $y = x$ with $x \in [0, 1]$. Hence, Γ is a compact set. Since f is continuous Weierstrass' theorem guarantees the existence of a solution. Obviously, the maximum is attained at $(x, y) = (1, 1)$.

We next use Algorithm 4.20 to solve the problem.

Step 1. $Df = [1 \ 2]$, $Dg_1 = [2x \ 2y]$, $Dg_2 = [-1 \ 0]$ and $Dh = [1 \ -1]$. All partial derivatives are continuous. So, $f, g, h \in C^1$.

Step 2. Matrix $\tilde{G}$ is given by

$$\tilde{G} = \begin{bmatrix} 2x & 2y & x^2 + y^2 - 2 & 0 \\ -1 & 0 & 0 & -x \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

To see whether this matrix is always full row rank on Γ , we use Gaussian elimination. One easily verifies that matrix $\tilde{G}$ has full row rank if and only if next matrix G has full row rank for all $(x, y) \in \Gamma$.

$$G := \begin{bmatrix} -1 & 0 & 0 & -x \\ 0 & 1 & 0 & -x \\ 0 & 0 & x^2 + y^2 - 2 & 2x(-x + y) \end{bmatrix}.$$

Clearly matrix G has not full row rank iff. simultaneously $x^2 + y^2 - 2 = 0$ and $2x(x - y) = 0$.

Points that satisfy these two equalities are: $(0, \pm\sqrt{2})$, $(1, 1)$ and $(-1, -1)$. One easily verifies that from these points only $(1, 1) \in \Gamma$. So, $(1, 1)$ is a candidate solution.

Step 3. The Lagrange function is:

$$L(x, y, \lambda_1, \lambda_2, \mu_1) = x + 2y - \lambda_1(x^2 + y^2 - 2) + \lambda_2x - \mu_1(x - y).$$

Step 4. The necessary conditions are:

1. $[1 - 2\lambda_1x + \lambda_2 - \mu_1 \ 2 - 2\lambda_1y + \mu_1] = [0 \ 0]$.
2. i) $\lambda_1(x^2 + y^2 - 2) = 0$; ii) $\lambda_2x = 0$.
3. i) $x^2 + y^2 \leq 2$; ii) $x \geq 0$; and iii) $\lambda_i \geq 0$, $i = 1, 2$.
4. $x - y = 0$.

Substitution of $x = y$ in item 2.i shows that $\lambda_1(x^2 - 1) = 0$. So either $\lambda = 1$ or $x = 1$ (since by item 3.ii, $x \geq 0$).

Case 1: $\lambda_1 = 1$. Item 1 reduces then to: $[1 + \lambda_2 - \mu_1 \ 2 + \mu_1] = [0 \ 0]$. This implies $\mu_1 = -2$ and $\lambda_2 = -3$, which violates item 3.iii.

Case 2: $x = 1$. By item 2.ii, $\lambda_2 = 0$. Therefore, item 1 reduces to $[1 - 2\lambda_1 - \mu_1 \quad 2 - 2\lambda_1 + \mu_1] = [0 \ 0]$.

This implies that $\lambda_1 = \frac{3}{4}$ and $\mu_1 = \frac{-1}{2}$. Hence, $(x, y) = (1, 1)$ is an appropriate solution.

Since there is just one candidate solution, by Weierstrass' theorem, the maximum is attained at this point $(1, 1)$.

EXERCISE 4.6 Solve the next constrained optimization problems.

- a. Maximize $f(x, y) = x - y$, subject to the constraints $y \geq -x$, $y \leq x$ and $x^2 + y^2 = 2$.
- b. Maximize $U(x, y, z) = x^\alpha y^\beta z^\gamma$ subject to the constraints $x \geq 15$, $y \geq 10$, $z \geq 5$, $p_1 x + p_2 y + p_3 z = 100$ and $x + y \leq 50$. Here, all parameters are positive, $\alpha + \beta + \gamma = 1$, $p_1 = 1$, $p_2 = 2$ and $p_3 = 3$. In particular evaluate the cases $\alpha = \beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, and $\alpha = \beta = 0.1$, $\gamma = 0.8$.

4.6 Solving the problem without explicit constraint qualifications

Theorem 4.15 provides a method for identifying candidate solutions to our constrained maximization problem (4.8), under the assumption that a regularity condition holds at these candidates. This regularity condition, or constraint qualification, is required in order to apply the Implicit Function Theorem at a candidate solution. Consequently, in Algorithm 4.20, in order to find all candidate solutions, we must also consider points where this constraint qualification fails.

In this section, we show that it is possible to bypass Step 2 of the algorithm by introducing an additional parameter. The advantage is that Algorithm 4.20 becomes simpler in appearance. However, including this additional parameter generally results in more candidate solutions satisfying the conditions of the modified Algorithm 4.20, which is clearly a disadvantage.

To motivate the adapted version of the Kuhn-Tucker Theorem 4.15, consider the simplest case of the problem: the two-variable, one-inequality-constraint problem discussed in Section 4.2. Specifically, we reconsider on an open set C of $\mathbb{R}^2$ the problem:

$$\text{Maximize } f(x, y), \text{ subject to } g(x, y) \leq b, \text{ where } f, g : C \rightarrow \mathbb{R}. \quad (4.14)$$

If the constraint qualification is satisfied at the optimal solution of this problem, we have shown that conditions 1–3 in Theorem 4.8 hold. Now, assume the constraint qualification is not met at the optimal solution. That is, $f(x, y)$ attains a maximum at (c, d) , where $g(c, d) = b$ and $Dg(c, d) = [0 \ 0]$. Consider the Lagrange function

$$L(x, y, \lambda_0, \lambda) := \lambda_0 f(x, y) - \lambda(g(x, y) - b). \quad (4.15)$$

Then, with $\lambda_0 = 0$ and λ any positive number, the next (in)equalities are satisfied: 1. $DL(c, d) = [0 \ 0]$; 2. $\lambda(g(c, d) - b) = 0$ and 3. $g(c, d) \leq 0$ and $\lambda > 0$.

Combining this observation with the result of Theorem 4.8 leads to the following theorem, formulated by Fritz John in 1948 (see [12]).

THEOREM 4.32 (*F. John*)

Consider the constrained maximization problem (4.14) with both $f, g \in C^1$. Assume the problem has a solution at (c, d) . Let $L(x, y, \lambda_0, \lambda) := \lambda_0 f(x, y) - \lambda(g(x, y) - b)$ be the with this problem associated Lagrange function. Then, there exists a $\lambda_0 \in \{0, 1\}$ and $\lambda \in \mathbb{R}$ such, that $(c, d, \lambda_0, \lambda)$ satisfy the next set of (in)equalities

1. $[D_1 L(c, d, \lambda) \ D_2 L(c, d, \lambda)] = [0 \ 0]$;
2. $\lambda(g(c, d) - b) = 0$;
3. $g(c, d) \leq b$ and $\lambda \geq 0$;
4. $(\lambda_0, \lambda) \neq (0, 0)$.

We illustrate this theorem in two examples. As the first example we reconsider Example 4.12.

EXAMPLE 4.33 Consider the constrained maximization problem

$$\text{maximize } f(x, y) = 2x^2 + y^2 + 4x \text{ subject to } g(x, y) = x^2 + y^2 \leq 4.$$

We already showed that both $f, g \in C^1$. To determine the candidate solutions, we consider the Lagrange function:

$$L(x, y, \lambda_0, \lambda) = \lambda_0(2x^2 + y^2 + 4x) - \lambda(x^2 + y^2 - 4).$$

If $\lambda_0 = 1$, it was shown in Example 4.12 that

$$(x, y, \lambda) = (-1, 0, 0), (x, y, \lambda) = (2, 0, \frac{3}{2}) \text{ and } (x, y, \lambda) = (-2, 0, \frac{1}{2})$$

are candidate solutions.

If $\lambda_0 = 0$, then by item 4 of Theorem 4.32, we must have $\lambda \neq 0$. Consequently, item 2 implies $x^2 + y^2 = 4$. Moreover, by item 1 we obtain $Dg(x, y) = [0 \ 0]$, that is, $[2x \ 2y] = [0 \ 0]$, which yields $(x, y) = (0, 0)$. Since $(0, 0)$ violates item 2, this candidate is discarded.

Hence, we obtain the same three candidate solution as in Example 4.12. Proceeding as in that example, we conclude that the maximum is attained at $(x, y) = (2, 0)$.

EXAMPLE 4.34 Consider the constrained optimization problem

$$\text{minimize } f(x, y) = x^2 + (y - 1)^2 \text{ subject to } g(x, y) = y^2 \leq x^4.$$

Clearly, both f, g belong to C^1 . Hence, to determine the candidate solutions, we may directly consider the associated Lagrange function:

$$L(x, y, \lambda_0, \lambda) = -\lambda_0(x^2 + (y - 1)^2) - \lambda(y^2 - x^4).$$

By Theorem 4.32, candidate optimal solutions must satisfy the following conditions:

1. $[D_1 L(c, d, \lambda) \ D_2 L(c, d, \lambda)] = [-2\lambda_0 x + 4\lambda x^3 \ -2\lambda_0(y - 1) - 2\lambda y] = [0 \ 0]$
2. $\lambda(y^2 - x^4) = 0$
3. $y^2 \leq x^4$ and $\lambda \geq 0$
4. $(\lambda_0, \lambda) \neq (0, 0)$, where $\lambda_0 \in \{0, 1\}$.

To solve this system of (in)equalities we distinguish two cases.

Case 1. $\lambda_0 = 1$.

From item 1 we obtain $x(1 - 2\lambda x^2) = 0$ and $(1 + \lambda)y - 1 = 0$.

If $\lambda = 0$, the solution is $(x, y) = (0, 1)$, which violates item 3. Hence, we may assume $\lambda > 0$, and by item 2 it follows that $y^2 = x^4$.

Next, consider the solution $(x, y) = (0, \frac{1}{1+\lambda})$, which satisfies item 1. Since $y^2 \neq x^4$, this point does not qualify as a candidate solution.

Finally, consider solutions of the form $(x^2, y) = (\frac{1}{2\lambda}, \frac{1}{1+\lambda})$, which also satisfy item 1. Substituting into $x^4 = y^2$ yields

$$\frac{1}{4\lambda^2} = \frac{1}{(1 + \lambda)^2},$$

or, equivalently, $(1 + \lambda)^2 = 4\lambda^2$. This gives $1 + \lambda = 2\lambda$ or $1 + \lambda = -2\lambda$. Since $\lambda > 0$, we conclude that

$$(x, y, \lambda_0, \lambda) = (\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, 1, 1)$$

are two candidate solutions. Note that $f(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}) = \frac{3}{4}$.

Case 2. $\lambda_0 = 0$.

By item 4, we must have $\lambda > 0$. In this case, $(x, y) = (0, 0)$ satisfies item 1. It is straightforwardly to verify that $(x, y, \lambda_0, \lambda) = (0, 0, 0, \lambda)$, for any $\lambda > 0$, also satisfies items 2-4, and thus constitutes another candidate solution.

Since $f(0, 0) = 1 > \frac{3}{4}$, the only remaining candidates for a global minimum location are

$$(x, y) = (\pm \frac{1}{\sqrt{2}}, \frac{1}{2}).$$

Note that the feasible set is connected, and that f is a continuous function bounded below by zero. Hence, by the continuity property theorem, Theorem 2.15, f admits an infimum strictly greater than

zero on the constrained set. Moreover, by Theorem 4.19, at the points $(x, y, \lambda) = (\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, 1)$, with

$$B^T := [1 \pm \sqrt{2}] \text{ and } H\tilde{L} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix},$$

we have $B^T H\tilde{L} B < 0$. Therefore, f has a local minimum at each of these points on the constrained set.

Finally, note that $f(x, y) > 1$ for all $x \geq 1$ and $x \leq -1$. By restricting the search for a minimum to the compact set $\{g(x, y)\} \cap \{x^2 \leq 1\}$, it follows that above mentioned local minimum locations are also global minimum locations of the problem.

The reader is asked to provide a proof for the general problem (4.1), in which the optimization problem involves more than two variables and/or more than one constraint, in Exercise 4.25. The exercise closely follows the structure of the proof given by Takayama in [28][Theorem 1.D.9].

Unfortunately, the result obtained for this case cannot be used to derive the necessary conditions for problem (4.11), where equality restrictions are also included in the description of the constraint. The trick of rewriting an equality restriction as two inequality restrictions does not resolve the problem in this case, since the with these two inequalities corresponding Lagrange parameters may both be strictly positive while their difference is zero. Consequently, one cannot conclude that at least one of the multipliers $\lambda_i \geq 0$ or μ_i is nonzero.

Readers interested in a proof that the theorem also in this case flies are referred to the literature.

THEOREM 4.35 (*F. John, general case*)

Consider the constrained maximization problem (4.11) where $f, g_i, h_j \in C^1$. Assume the problem has a solution at $s = a$. Let $L(s, \lambda_0, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m) :=$

$$\lambda_0 f(s) - \lambda_1 (g_1(s) - b_1) - \dots - \lambda_k (g_k(s) - b_k) - \mu_1 (h_1(s) - c_1) - \dots - \mu_m (h_m(s) - c_m) \quad (4.16)$$

be the with this problem associated Lagrange function.

Then, there exists a $\lambda_0 \in \{0, 1\}$ and $\lambda_i \geq 0, i = 1, \dots, k, \mu_j \in \mathbb{R}, j = 1, \dots, m$, such that, $(a, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m)$ satisfy the next set of (in)equalities

1. $[D_1 L(a, \lambda_0, \lambda_1, \dots, \lambda_m) \dots D_n L(a, \lambda_0, \lambda_1, \dots, \lambda_m)] = [0 \dots 0]$.
2. $\lambda_i (g_i(a) - b_i) = 0, i = 1, \dots, k$.
3. $g_i(a) \leq b_i, h_i(a) = c_i$.
4. At least one of the parameters $\lambda_0, \lambda_i, i = 1, \dots, k, \mu_j, j = 1, \dots, m$, differs from zero.

EXAMPLE 4.36 (*peak load pricing, see [27][p.540] for a generalization of this example*)

Consider an electricity company who sells an amount of electric power x_i in period i at a price p_i , $i = l, h$ (l is off-peak period, h is peak period). The total capacity the company can produce at any period is k . The operating cost over both periods is $C(x_l, x_h)$ and the cost of maintaining output

capacity at level k is $D(k)$, where both functions are continuous differentiable. The company is then faced with maximizing its total profits

$$\pi(x_l, x_h, k) = p_l x_l + p_h x_h - C(x_l, x_h) - D(k) \text{ subject to } 0 \leq x_i \leq k, i = l, h.$$

We assume, moreover, that beyond a certain threshold of capacity the cost of maintaining the output at such high levels is not profitable.

To solve the problem we first note that both the profit function and the constraints $g_i(x_l, x_h, k) = x_i - k$ and $\tilde{g}_i(x_l, x_h, k) = -x_i$, $i = l, h$, are continuous differentiable. So, we can use Theorem 4.35 to find the candidate optimal solutions. To that end we introduce the Lagrange function

$$L(s, \lambda_0, \lambda_1, \dots, \lambda_4) := \lambda_0 \pi(x_l, x_h, k) - \lambda_1(x_l - k) - \lambda_2(x_h - k) + \lambda_3 x_l + \lambda_4 x_h.$$

Item 1. of this theorem yields the next necessary conditions:

$$\lambda_0(p_l - D_1 C) - \lambda_1 + \lambda_3 = 0; \lambda_0(p_h - D_2 C) - \lambda_2 + \lambda_4 = 0; -\lambda_0 D' + \lambda_1 + \lambda_2 = 0.$$

Since λ_i must be nonnegative we directly conclude from this that, in case $\lambda_0 = 0$, there is no solution. This, since it follows that $\lambda_i = 0$, $i = 0, \dots, 4$, which violates item 4.

So, $\lambda_0 = 1$. Since, by assumption, the output capacity is restricted the domain of this optimization problem is compact. Since the profit function is continuous, it follows by Weierstrass' theorem that a solution exists. By calculating all solutions of the set of (in)equalities resulting from items 1-3, and evaluating the profit at these solutions one can determine then the optimal one. Of course, without an explicit formulation of the involved functions this exercise can not be explicitly performed. However, we will not go that far here and just assume that the functions are such that the optimal solution is attained at a capacity level k^* and production levels $x_i^* > 0$ such that only at the high production level capacity is binding. So, the necessary conditions from items 1-3 reduce to

$$p_l - D_1 C = 0; p_h - D_2 C - \lambda_2 = 0; -D' + \lambda_2 = 0; x_h = k^*; \lambda_2 \geq 0.$$

So it follows that at such an optimum necessarily i) $D'(k^*) \geq 0$; ii) marginal operating cost during the off-peak period equals the charged price; and iii) during the high-peak period the charged price equals the sum of marginal operating cost and marginal capacity cost. Or, stated differently, from iii) it follows that at high-peak periods the charged price should include a mark-up to account for the capacity cost.

EXERCISE 4.7 Solve the next constrained optimization problems.

- Maximize $U(x, y) = x + \ln(1 + y)$, subject to the constraints $x + 2y \leq 4$, $x \geq 0$ and $y \geq 0$.
- Minimize $C(K, L) = rK + \omega L$ subject to the constraints $P(K, L) = 10$ where $P(K, L) = \sqrt{K} + \sqrt{L}$; $L \geq 16$, $K \leq 81$ and $\frac{K}{L} \geq 2$.

- c. Maximize $U(\mu, \sigma) = \mu - \frac{1}{2}\sigma^2$ w.r.t. ω_1 and ω_2 , subject to the constraints $\mu = 2\omega_1 + 3\omega_2$, $\sigma = \omega_1 + \omega_2$ where $\omega_i \geq 0$ and $\omega_1 + \omega_2 \leq 1$.
- d. Maximize $U(x, y) = x^{\frac{1}{2}}y^{\frac{1}{4}}$, subject to the constraints $4x + 3y = I$, where $I > 0$ is some fixed number, $x \geq \frac{1}{8}I$ and $y \geq \frac{1}{10}I$.

4.7 A geometric approach to solve the problem

Up to this point, we have addressed constrained optimization problems by explicitly or implicitly solving the binding constraints at a solution $x = a$ for m_1 variables in terms of the remaining variables, at least locally. Substituting these m_1 variables then reduces the problem to an unconstrained optimization problem in fewer variables. A drawback of this approach is that it is not always possible to solve the m_1 equations for m_1 variables as functions of the remaining variables.

An alternative approach, commonly used in the literature and which (at least theoretically) avoids this difficulty, is to consider, at the point $x = a$, the set of all feasible linear directions along which one can move while remaining within the constraint set S . That is, we consider all directions u at $x = a$ for which there exists a continuously differentiable function $s(t)$ taking values in S on some interval $[0, t_1)$, with $s(0) = a$, such that, in a neighborhood of $t = 0$, the tangent line to $s(t)$ converges to the line $r(t) = a + ut$. Roughly speaking, by evaluating f along all such directions u , one can determine whether f attains a maximum at $x = a$. In particular, when f is differentiable, this requires that along every feasible direction the corresponding restriction of f has a non-increasing derivative at $x = a$. Equivalently, the directional derivative of f at $x = a$ in any feasible direction u must satisfy $D_u f(a) = Df(a)u \leq 0$.

Below, we formalize the approach outlined above. Since the purpose of this section is not to provide an in-depth analysis of this method, the reader is referred, for example, to [18][Section 12.3] or [11][Sections 1.8–1.10] for a rigorous treatment and proofs of the results discussed below.

DEFINITION 4.37 Let $a \in S \subset \mathbb{R}^n$. A vector $u \in \mathbb{R}^n$ is called a *tangent* to S at a if there exists a continuous differentiable function $s(t) : [0, \epsilon) \rightarrow S$ for some $\epsilon > 0$, such that¹³

$$s(t) = a + ut + o(t).$$

The closure¹⁴ of the set of all these tangent vectors u is called the *tangent cone* to S at a and is denoted by $T_S(a)$.

¹³ $o(t)$ means here that $\frac{s(t)}{t}$ converges to zero if t converges to zero.

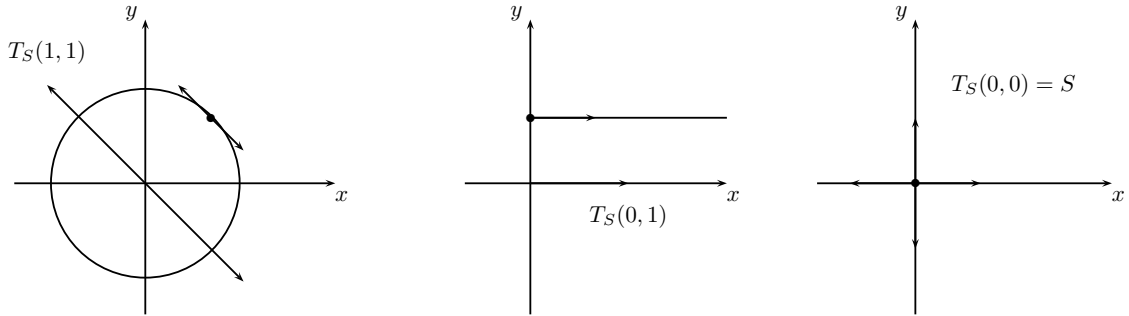
¹⁴ The *closure* of a set S is the set containing S and all of its accumulation points.

REMARK 4.38

1. From above definition it is clear that always $0 \in T_S(a)$ and, with $u \in T_S(a)$, also $\lambda u \in T_S(a)$, for all $\lambda > 0$. So, $T_S(a)$ consists indeed of a closed set of vectors that constitutes a cone.
2. Notice that if a is an interior point of S , all vectors in $\mathbb{R}^n$ belong to the tangent cone $T_S(a)$.

EXAMPLE 4.39

1. In Figure 4.12 we illustrate the tangent cone for 3 cases. In panel a), the set S is given by all points on the circle $x^2 + y^2 = 2$. At $a = (1, 1)$ the tangent cone coincides with all vectors $\lambda[1 \ -1]^T$, $\lambda \in \mathbb{R}$. For, if we approach a along the circle from below, $[1 \ -1]^T$ is a tangent vector. Whereas, if we approach a along the circle from above, $[-1 \ 1]^T$ is a tangent vector.
- In panel b), the set S is given by the positive part of the line $y = 1$ in $\mathbb{R}^2$. At $(0, 1)$, $[1, 0]^T$ is a tangent vector. The tangent cone $T_S(a) = \{ \lambda[1 \ 0]^T, \lambda \geq 0 \}$.
- In panel c), $S = \{(x, y) \mid xy = 0\}$. In this case one easily verifies that the tangent cone at $a = (0, 0)$, $T_S(0, 0)$ coincides with the set S .
2. In Figure 4.13 we illustrate in item a) the tangent cone if $S = \{(x, y) \mid (x-1)^2 + (y-2)^2 \leq 1, y \geq 1\}$. In this case $T_S(0, 1)$ consists of all vectors pointing into the positive orthant, as indicated in panel b).



(a) $T_S(1, 1)$; $S = \{(x, y) \mid x^2 + y^2 = 2\}$. (b) $T_S(0, 1)$; $S = \{(x, 1) \mid x \geq 0\}$. (c) $T_S(0, 0)$; $S = \{(x, y) \mid xy = 0\}$.

FIGURE 4.12 – Examples of tangent cones.

Following the above mentioned arguments we have then next result.

THEOREM 4.40 Consider the mixed constrained maximization problem (4.11) on the constraint set D . Assume the problem has a solution at $s = a$. Then $Df(a)u \leq 0$ for all $u \in T_D(a)$.

A difficulty with this approach is that, in general, it is analytically challenging to determine all feasible directions u in the tangent cone for an arbitrary set S . In our setting, however, the constraint set S is defined as the intersection of sets, each characterized as the locus of points at which a given contin-

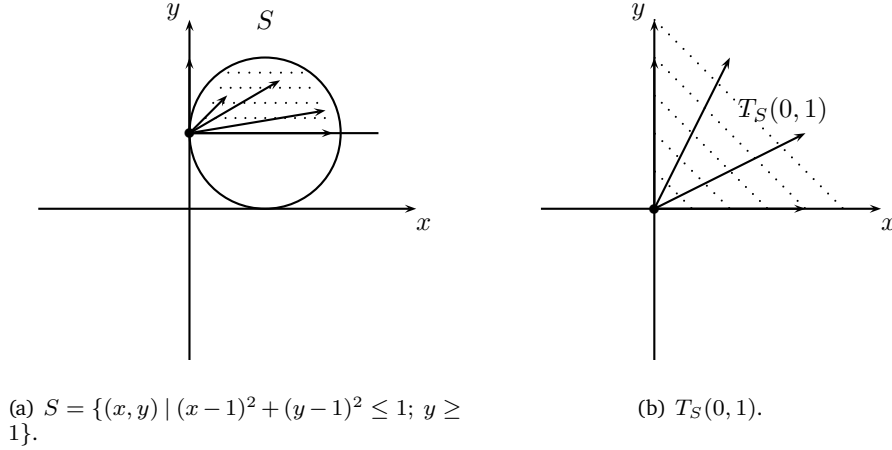


FIGURE 4.13 – Tangent cone $S = \{(x, y) \mid (x-1)^2 + (y-2)^2 \leq 1, y \geq 1\}$ at $(0, 1)$.

uously differentiable function satisfies either an equality or an inequality constraint. This structure makes it possible to obtain a more explicit description of the tangent cone at a point $s = a$.

The idea is to linearize the constraint set at $s = a$ and then determine which directions u are feasible within this linearized constraint set. Recall that if a constraint $g_i(a)$ is non-binding at $s = a$, the tangent cone coincides with the entire space. Therefore, to derive this more explicit characterization of the tangent cone at $s = a$, it suffices to consider only those constraints that are binding at that point.

Now assume that the constraint $g_1(x) \leq 0$ is binding at $s = a$, and let u be a tangent direction of the set $S := \{x \mid g_1(x) \leq 0\}$ at $s = a$. By definition, there then exists a continuously differentiable curve $x(t) = a + tu + o(t)$ contained in S . Applying the chain rule yields that the derivative $\frac{dg_1(x(t))}{dt}$ at $t = 0$ satisfies $Dg_1(a)u \leq 0$.

In other words, every vector u in the tangent cone is contained in the set

$$TL_S(a) := \{v \mid Dg_1(a)v \leq 0\}.$$

Ideally, one would hope that every vector in $TL_S(a)$ also belongs to the tangent cone of S at $s = a$. Unfortunately, this is not always the case, as will be demonstrated in the next example.

The good news, however, is that in many situations the two sets do coincide. For instance, if the Implicit Function Theorem applies to $g_1(x)$ at $s = a$, then this approach is valid, and the tangent cone at $s = a$ coincides with the set of all vectors u satisfying $Dg_1(a)u \leq 0$.

Similarly, if the constraint set is defined by an equality constraint $h_1(x) = 0$, then the set of all directions u at $s = a$ that remain within the set $h_1(x) = 0$ is typically given by all vectors satisfying $Dh_1(a)u = 0$. These observations motivate the introduction of the so-called set of *linearized feasible directions* at $s = a$ for the constraint set D , which we denote by $TL_D(a)$.

DEFINITION 4.41 Consider the constrained maximization problem (4.11) where $g_i, h_j \in C^1$. Let $a \in D$. Assume $g_i(a)$, $i = 1, \dots, k_1$, are binding and the other constraints $g_j(a)$, $j = k_1 + 1, \dots, k$, are non-binding. Then

$$TL_D(a) := \{u \in \mathbb{R}^n \mid Dh_i(a)u = 0, i = 1, \dots, m, \text{ and } Dg_i(a)u \leq 0, i = 1, \dots, k_1, \}$$

is called the set of *linearized feasible directions* to D at $s = a$.

Similar to Remark 4.38, item 1, one can show that $TL_D(a)$ is a cone. As argued above, this leads to the following result.

THEOREM 4.42 Let $a \in D$. Then $T_D(a) \subset TL_D(a)$.

That both sets not always coincide demonstrates next example.

EXAMPLE 4.43

1. Consider the set $S := \{x \in \mathbb{R} \mid g(x) = x^3 \leq 0\}$. Then $T_S(0) = \{u \in \mathbb{R} \mid u \leq 0\}$, whereas $TL_S(0) = \{u \in \mathbb{R} \mid 3x^2u \leq 0, \text{ at } x = 0\} = \mathbb{R}$.
2. Consider the set by all points located on the unit circle, i.e., $S = \{(x, y) \mid h(x, y) = (x^2 + y^2 - 1)^2 = 0\}$. Then at any point $a = (a_1, a_2) \in S$, $T_S(a) = \{u \mid u = \lambda[-a_2 \ a_1]^T, \lambda \in \mathbb{R}\}$. Whereas, at any point a , $TL_S(a) = \{u \in \mathbb{R}^2 \mid 4(a_1^2 + a_2^2)[a_1 \ a_2]u \leq 0\} = \{u \in \mathbb{R}^2 \mid [0 \ 0]u \leq 0\}$, which set consists of all vectors in $\mathbb{R}^2$.

Note that in both items of the example above, the derivative of the function is zero at the point(s) under consideration; that is, $g'(0) = 0$ and $h'(a) = [0 \ 0]$. In particular, in the second item this implies that the conditions for applying the Implicit Function Theorem are not satisfied at any point a .

In the literature, any condition that ensures that the sets $T_D(a)$ and $TL_D(a)$ coincide is referred to as a *constraint qualification*. Below, we summarize some well-known constraint qualifications.

THEOREM 4.44 Let $a \in D$. Assume $g_i(a)$, $i = 1, \dots, k_1$, are binding and the other constraints $g_j(a)$, $j = k_1 + 1, \dots, k$, are non-binding. Consider next conditions:

1. With $\tilde{g}_1(x) := [g_1(x) \cdots g_{k_1}(x) \ h_1(x) \cdots h_m(x)]^T$, $D\tilde{g}_1(a)$ has full row rank $k_1 + m$. (The linear independence constraint qualification).
2. $g_i(x)$, $i = 1, \dots, k_1$ are convex and $h_i(x)$, $i = 1, \dots, m$ are affine functions (i.e. $h_i(x) = a_i^T x + d_i$, $a_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$). Furthermore, there exists a point a such that $g_i(a) < b_i$, $i = 1, \dots, k_1$ and $h_i(a) = c_i$, $i = 1, \dots, m$. (The Slater's constraint qualification for convex problems.)
3. Either $h_i(x)$ are affine functions or $Dh(a)$ has full row rank and there exists $d \in \mathbb{R}^n$ such that $Dg_i(a)d < 0$, $i = 1, \dots, k_1$, and $Dh(a)d = 0$. (The Mangasarian-Fromovitz constraint qualification.)

If either one of above three conditions is satisfied then $T_a(D) = TL_a(D)$.

That the constraint qualifications mentioned above are merely sufficient conditions for the coincidence of the tangent cone and the linearized tangent cone is illustrated in the next example. In Exercise 5.18, the reader is asked to show that, for convex programming problems, Slater's constraint qualification is indeed sufficient to conclude that the maximizer satisfies the (KKT) conditions.

EXAMPLE 4.45 Consider the constraint set $S := \{(x, y) \in \mathbb{R}^2 \mid g_1(x, y) := y - x^2 \leq 0 \text{ and } g_2(x, y) := -y - x^2 \leq 0\}$. Then $TL_S(0, 0) = \{u \in \mathbb{R}^2 \mid [-2x \ 1]u \leq 0 \text{ and } [-2x \ -1]u \leq 0 \text{ at } (x, y) = (0, 0)\}$. Whereas $T_S(0, 0)$ is visualized in Figure 4.14. We observe from this that $T_S(0, 0) = TL_S(0, 0) = \{u \in \mathbb{R}^2 \mid u_2 = 0\}$.

Notice that $Dg_1(0, 0) = [0 \ 1]$ and $Dg_2(0, 0) = [0 \ -1]$. From this one easily verifies that as well condition 1 and 3 in Theorem 4.44 are not satisfied. Furthermore, g_1 is a convex function whereas g_2 is a (nonlinear) concave function. Thus, the conditions listed in item 2 of this theorem do not hold too. In other words, none of the constraint qualifications are satisfied, yet the tangent cone and the linearized tangent cone still coincide.

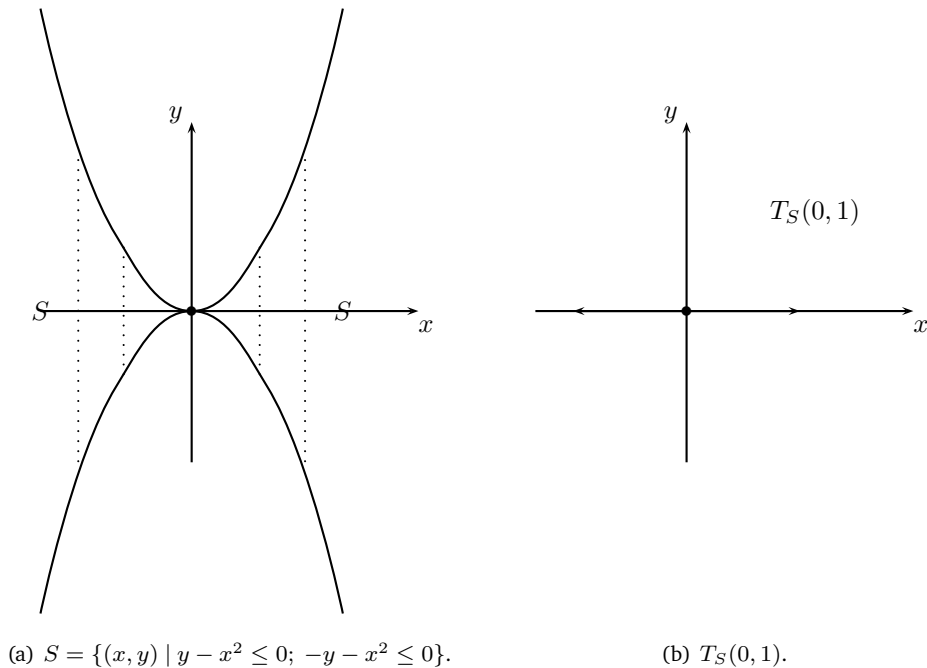


FIGURE 4.14 – Tangent cone $S = \{(x, y) \mid y - x^2 \leq 0, -y - x^2 \leq 0\}$ at $(0, 0)$.

EXERCISE 4.8

Use Theorem 4.44 to determine the tangent cone for next sets at the indicated point a .

- a. $S = \{(x, y, z) \mid |x + y| \leq 1; x^2 + y^2 + z^2 \leq 1; x + z \leq 1 \text{ and } |y + z| \leq 1\}$ at $(0, 1, 0)$.
- b. $S = \{(x, y) \mid x^2 + (y - 1)^2 \leq 2; x^2 - y^2 \leq 1 \text{ and } x \leq 1\}$ at $(1, 0)$.
- c. $S = \{(x, y) \mid x^2 + y^2 \leq 1; (x - 1)^2 + y^2 \leq 2; \text{ and } x - y \leq -1\}$ at $(0, 1)$.

Using Farkas' lemma¹⁵ [9], a well-known result in linear algebra, the reader is asked in Exercise 4.26 to show that when the tangent cone and the linearized tangent cone coincide, the (KKT) conditions are recovered as necessary first-order conditions, provided that the constrained optimization problem (4.11) has a solution.

THEOREM 4.46 *Consider the mixed constrained maximization problem (4.11) on the constraint set D . Assume the problem has a solution at $s = a$. Assume $T_D(a) = TL_D(a)$. Then at $s = a$ there exist $\lambda_i, \mu_i \in \mathbb{R}$ such that the conditions (KKT) apply.*

REMARK 4.47 In particular, we conclude from Theorems 4.46 and 4.44 that, if item 2 of Theorem 4.44 applies, Step 2 in Algorithm 4.20 can be omitted when solving the constrained optimization problem (4.8).

We conclude this section by noting that, using this approach, similar second-order conditions for the existence of a local maximum can be obtained, as derived earlier in this chapter. Nocedal et al. [18][Theorems 12.5 and 12.6] show that if (a, λ, μ) is a solution of the KKT conditions, then the constrained optimization problem has a strict local maximum at $s = a$ provided that the Hessian of the Lagrange function is negative definite in all directions within the tangent space along which the objective function does not vary to first-order.

Below, we state this slightly more general result. Exercise 4.10, items e and f, illustrate its added value.

THEOREM 4.48 *Consider the mixed constrained maximization problem (4.11) on the constraint set D .*

1. *Assume the problem has a local maximum at $s = a$ and the linear independence constraint qualification is met at this point. Let (a, λ, μ) satisfy the (KKT) conditions. Then, $v^T HL(a, \lambda, \mu)v \leq 0$ for all $v \in TL_D(a)$.*
2. *Assume (a, λ, μ) satisfy the (KKT) conditions and $v^T HL(a, \lambda, \mu)v < 0$ for all $v \in TL_D(a)$, $v \neq 0$. Then the problem has a local maximum at $s = a$.*

¹⁵ Farkas was a Hungarian mathematician who lived from 1847–1930

Mixed Exercises

EXERCISE 4.9 Use the theory presented in this chapter to solve next constrained optimization problems.

- Maximize $f(x, y) = x^3 + y^3$ subject to $|x| \leq 1$ and $|y| \leq 1$.
- Maximize $f(x, y) = x + y$ subject to $|xy| \leq 1$.
- Maximize $f(x_1, x_2) = x_1 x_2$ subject to $g(x_1, x_2) = x_1 + 8x_2 - 4 \geq 0$ and $h(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$.
- Minimize $f(x) = \sqrt{1 - x^2}$, $x \in [-1, 1]$ subject to $g(x) = x \leq 1$.
- Minimize $f(x, y) = (x + 2)^2 + y^2$ subject to $y \geq |x|$. Hint: first rewrite the constrained set as the intersection of two sets $\Gamma_i := \{(x, y) \mid g_i(x, y) \leq 0\}$, where the functions g_i are differentiable.
- Minimize $f(x, y) = e^{x+y}$ subject to $x^2 \geq 1$ and $y^2 \geq 1$.
- Minimize $f(x, y) = e^{x+y^2}$ subject to $x^2 \leq 1$ and $y^2 \geq 1$.

EXERCISE 4.10 Solve next constrained optimization problems.

- Maximize $f(x, y, z) = x \ln(xyz) - x - y - z$ s.t. $xyz \geq \frac{1}{2}$.
- Maximize $f(x, y, z) = x^2 - y^2 + z^2$ s.t. $x^2 + y^2 + z^2 \leq 3$ and $xyz = 1$.
- Find the point $(x, y) \in \mathbb{R}^2$ that has a minimal distance to the disc, with center $(0, 0)$ and radius $\sqrt{2}$, and the hyperbola $cd = 9$. That is, minimize $(x - a)^2 + (y - b)^2 + (x - c)^2 + (y - d)^2$ w.r.t. x, y, a, b, c and d s.t. $a^2 + b^2 \leq 2$ and $cd = 9$.
- Maximize $f(x, y, z) = 14(x + y + z) - xy - xz - yz$ s.t. $x \leq 0$, $x + y = 2$ and $y + z^2 = 5$.
- Minimize $f(x, y, z) = xy + xz + yz - 3x - 3y - 3z$ s.t. $x \leq 0$, $x - y = -1$ and $y + z^2 = 2$.
- Minimize $f(x, y, z) = x^2 - xy - yz + z^2$ s.t. $x \geq 0$, $y \geq 0$, $z \geq 0$ and $y \leq x^2 + z^2$.
- Maximize $f(x, y, z) = x^3 + y^3 + z^3$ s.t. $x^2 + y^2 \leq 1$ and $y^2 + z^2 \leq 1$.
- Minimize $f(x, y, z) = x^2 + y^2 - z^2$ s.t. $x + y \geq 1$ and $y^2 + z^2 \leq 1$.

EXERCISE 4.11 Find all local maxima of next constrained optimization problems.

- Optimize $f(x, y, z) = -6x + y + z$ s.t. $y \leq x^3$ and $z \leq x^3$.
- Optimize $f(x, y, z) = xyz$ s.t. $y \leq x^3$ and $z \leq x^3$.

EXERCISE 4.12

Determine the tangent and linearized cone for next sets at the indicated point a , respectively.

- $S = \{(x, y) \mid |x| \leq 1 \text{ and } |y| \leq 1\}$ at $(1, 1)$.
- $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$ at $(1, 0)$.
- $S = \{(x, y) \mid x^2 - y^2 \leq 0\}$ at $(0, 0)$.
- $S = \{(x, y, z) \mid \log(1 + x^2 + y^2 + z^2) \leq \log(4); \frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z} \geq 1; x + y + z \leq 3; x \geq 1; |z| \leq 2\}$ at $(0, 0)$.

Advanced, Deepening and Broadening Exercises

EXERCISE 4.13 (Reformulating optimization problems into the standard framework)

In this exercise a number of optimization problems are considered that can be reformulated into the standard framework.

Reformulate next problems as a (mixed) constrained maximization problem (4.11).

- Maximize $f(x)$ subject to $\max_x \{g_1(x), g_2(x)\} \leq a$.
- Maximize $f(x)$ subject to $|g_1(x)| + |g_2(x)| \leq a$.
- Minimize $f(x)$ where $f(x) = \max_x \{g_1(x), g_2(x)\}$.
- Minimize the Euclidean distance between two sets given by $S_i = \{x \mid g_i(x) \leq b_i\}$, $i = 1, 2$. Comment on this.

EXERCISE 4.14 (Solving an inequality constrained optimization problem as an equality constrained optimization problem)

In this exercise you are asked to show that an optimization problem subject to inequality constraints can be solved also as an equality constrained optimization problem by the introduction of so-called *slack variables*. For simplicity you are asked to prove this result if there is only one inequality restriction.

Assume $f(x), g(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous on an open set $S \subset \mathbb{R}^n$. Consider the set $C := \{x \in \mathbb{R}^n \mid g(x) \leq b, x \in S\}$ and $C_1 := \{(x, s) \in \mathbb{R}^{n+1} \mid g(x) + s^2 = b, x \in S\}$. Assume $a \in C$. Let $f_1(x, s) := f(x)$ for all s .

- Show that $s(x) := \sqrt{b - g(x)}$ is a continuous function on C .
- Assume $f(x)$ has a local maximum at $x = a$ on C . Show that $f_1(x, s)$ has a local maximum at $(x, s) = (a, \sqrt{b - g(a)})$ on C_1 .
- Assume $f_1(x, s)$ has a local maximum at $(x, s) = (a, \sqrt{b - g(a)})$ on C_1 . Show that $f(x)$ has a local maximum at $x = a$ on C .
- Is $(x, s) = (a, \sqrt{b - g(a)})$ on C_1 the only location where $f_1(x, s) = f(a)$? Comment on this.

EXERCISE 4.15 (Proof Lemma 4.7)

Assume $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous differentiable on an open set $S \subset \mathbb{R}^n$ containing a . Consider the set $C = \{x \in \mathbb{R}^n \mid g(x) \leq b\}$ and assume $g(a) = b$. Prove that

- $Dg_u(a)u \geq 0$ for any direction u pointing into C . Hint: Consider $\tilde{g}(t) = g(a + tu)$ for appropriate $t \geq 0$.
- $Dg_u(a)u \leq 0$ for any direction u pointing outside C .
- $Dg(a)$ is perpendicular to the tangent plane of the graph of $g(x)$ at $x = a$.

EXERCISE 4.16 (*Proof Theorem 4.19*)

Consider the assumptions and notation in Theorem 4.19.

- Show that $f(a)$ will be a local maximum of the constrained optimization problem to maximize $f(x)$ subject to the constraint set defined by $S_c = \{x \in \mathbb{R}^n \mid g_1(x) = \tilde{b}_1\}$, where $\tilde{b}_1 = [b_1 \cdots b_{m_1}]^T$, if the advertized constrained Hessian matrix is negative definite.
- Use item a. and Corollary 3.39 to show that a necessary condition for existence of a local maximum at a for the constraint optimization problem (4.8) is that $d(v) = v^T H \tilde{L}(a)v \leq 0$ for all $v \in \{v \mid Dg_1(a)v = 0\}$.
- Show that the constraint optimization problem (4.8) has a maximum at $x = a$ under the stated assumptions. Hint: Introduce slack variables into the problem (see Exercise 4.14) to rewrite the problem as an equality constraint problem. Next, use Theorem 3.38 to conclude.

EXERCISE 4.17 (*Reformulation Chebyshev distance problem as a Linear Programming Problem*)

The Chebyshev distance between 2 vectors x and y is defined as the maximum number of the absolute difference between entries of both vectors, i.e. $\max_i |x_i - y_i|$. In this exercise you are asked to show that the problem to find a vector in a convex polyhedron S that has a minimal Chebyshev distance to a given vector $c \in \mathbb{R}^n$ (not in S) can be reformulated as a linear programming problem.

- Let $t \geq 0$. Show that $|x_i - y_i| \leq t$ iff. $x_i - y_i \leq t$ and $x_i - y_i \geq -t$.
- Show that $\max_i |x_i - y_i| \leq t$ iff $x_i - t \leq y_i$ and $-x_i - t \leq -y_i$, $i = 1, \dots, n$.
- Assume the convex polyhedron S is described by the intersection of the half spaces $a_i^T x \leq b_i$. Show that the problem to find $\min_{s \in S} \max_i |s_i - c_i|$ (with $c \notin S$) is equivalent with the linear program to find $\max_{x,t} -t$ subject to the inequality constraints $a_i^T x - t \leq b_i + c_i$ and $-a_i^T x - t \leq -b_i - c_i$, $i = 1, \dots, n$.
- Consider 3 data points $(x_1, y_1) = (0, -1)$, $(x_2, y_2) = (1, 2)$ and $(x_3, y_3) = (2, 3)$. Determine parameters α and β that make the line $y = \alpha + \beta x$ as "close" to the points as possible in the sense of Chebyshev's distance. Verify your result.

EXERCISE 4.18 (*Optimization problem equivalence under change of variables*)

Consider the maximization of $f(x)$ subject to the constraints $g_i(x) \leq b_i$, $i = 1, \dots, k$, and $h_j(x) = c_j$, $j = 1, \dots, m$, on a set $C \subset \mathbb{R}^n$.

Assume $\phi(x) = [\phi_1(x) \cdots \phi_n(x)]^T$ is a mapping from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$ which maps no two elements of its domain to a single value in its range (*one-to-one mapping*). Moreover assume $\phi(x)$ is such that $C \subset \phi(D)$.

- Show that above optimization problem has a solution at x^* iff. with $\tilde{f}(z) := f(\phi(z))$, $\tilde{g}_i(z) := g_i(\phi(z))$ and $\tilde{h}_j(z) := h_j(\phi(z))$, the maximization of $\tilde{f}(z)$ subject to the constraints $\tilde{g}_i(z) \leq b_i$, $i = 1, \dots, k$, and $\tilde{h}_j(z) = c_j$, $j = 1, \dots, m$, has a solution at some point z^* . What is the relationship between the solution of both problems and between x^* and z^* ?
- Use the approach sketched in item a. to find the maximum of $f(x_1, x_2) = ax_1^\alpha + \ln(x_2)$ subject to $x_1^\alpha \ln(x_2) \leq 4$. Here a , α and x_i are strictly positive numbers.

EXERCISE 4.19 (Geometric Programming)

In this exercise we consider the so-called Geometric programming problem. In this problem objective and constraint functions are described by so-called posynomial functions. Using a transformation this problem can be rewritten as a convex optimization problem. You are asked to complete the details to reformulate this geometric programming problem as a convex optimization problem. Extensions and more information on this problem can be found in, e.g., [5][Chapter 4.5].

Consider the *monomial* $p_i(x) := a_i x_1^{\alpha_{1i}} * \cdots * x_n^{\alpha_{ni}}$, where $a_i > 0$, $x_i > 0$ and $\alpha_{ij} \in \mathbb{R}$. Then $f(x) := \sum_{i=1}^k p_i(x)$ is called a *posynomial*.

Let $f(x)$ and $g_i(x)$ be posynomials and b_i strictly positive numbers. The geometric programming problem is defined as:

$$\min_x f(x), \text{ subject to the constraints } g_i(x) \leq b_i.$$

- Let $y_i := \ln(x_i)$. Use this transformation (see Exercise 4.18), to formulate the corresponding equivalent optimization problem of maximizing $\tilde{f}(y) := -\sum_{i=1}^k \tilde{p}_i(y)$ subject to $\tilde{g}_i(y) \leq b_i$. Here, with $\alpha_i := [\alpha_{1i} \cdots \alpha_{ni}]^T$ and $\beta_i := \ln(a_i)$, $\tilde{p}_i(y) = e^{\alpha_i^T y + \beta_i}$.
- Show that $D\tilde{p}_i(y) = \alpha_i^T \tilde{p}_i(y)$ and $H\tilde{p}_i(y) = \alpha_i \alpha_i^T \tilde{p}_i(y)$.
- Show that $\tilde{f}(y)$ is a concave function and $\tilde{g}_i(y)$ are convex functions.
- Use above approach to find the maximum of $f(x_1, x_2) = \frac{x_2^2 + 2x_1x_2 + 1}{x_1^2 x_2^2}$ subject to $x_i \geq 1$, $i = 1, 2$. You may use a result from next chapter that the necessary KKT conditions are in this case sufficient too.

EXERCISE 4.20 (Linear-fractional programming)

Consider the problem to maximize the ratio of two affine functions over a constraint set described by linear inequalities. In this exercise you are asked to show that this problem can be reformulated as a linear programming problem.

Let $f(x) = \frac{a_1^T x + d_1}{a_2^T x + d_2}$ and $g_j(x) = c_j^T x$, $j = 1, \dots, k$, where $a_i, c_j \in \mathbb{R}^n$ and $d_i, b_j \in \mathbb{R}$, $i = 1, 2$; $j = 1, \dots, k$.

Then, the linear-fractional optimization problem is formalized as the maximization of $f(x)$ on the nonempty set defined by $C = \{x \in \mathbb{R}^n \mid g_j(x) \leq b_j, j = 1, \dots, k, \text{ and } a_2^T x + d_2 > 0\}$.

- Show that if $\hat{x}$ is a solution for the linear-fractional optimization problem, with $\hat{y} := \frac{\hat{x}}{a_2^T \hat{x} + d_2}$ and $\hat{z} := \frac{1}{a_2^T \hat{x} + d_2}$, $(\hat{y}, \hat{z})$ is a solution to the quasi linear programming problem:
 $\max_{y,z} a_1^T y + d_1 z$ subject to $c_j^T y - b_j z \leq 0$, $j = 1, \dots, k$, $a_2^T y + d_2 z = 1$ and $z > 0$.
- Show that if $(\hat{y}, \hat{z})$ is a solution to the quasi linear programming problem:
 $\max_{y,z} a_1^T y + d_1 z$ subject to $c_j^T y - b_j z \leq 0$, $j = 1, \dots, k$, $a_2^T y + d_2 z = 1$ and $z > 0$, the linear-fractional programming problem has the solution $\hat{x} = \frac{\hat{y}}{\hat{z}}$.
- Let $(\hat{y}, \hat{z})$ be a solution to the quasi linear programming problem:
 $\max_{y,z} a_1^T y + d_1 z$ subject to $c_j^T y - b_j z \leq 0$, $j = 1, \dots, k$, $a_2^T y + d_2 z = 1$ and $z > 0$.
 Show that $(\hat{y}, \hat{z})$ also solves the linear programming problem:

$\max_{y,z} a_1^T y + d_1 z$ subject to $c_j^T y - b_j z \leq 0$, $j = 1, \dots, k$, $a_2^T y + d_2 z = 1$ and $z \geq 0$. Hint: show this by assuming this is not the case.

d. Assume the constrained set in the linear-fractional programming problem is bounded.

Show that the linear-fractional programming problem has a solution iff. the linear programming problem formulated in item c. has a solution.

Show that if, moreover, in this case $d_2 \neq 0$ the solution of this linear programming problem can be found by determining the maximum w.r.t. y of the linear program $(a_1^T - \frac{d_1}{d_2} a_2^T) y + \frac{d_1}{d_2}$ subject to the constraints $\frac{1}{d_2} (1 - a_2^T y) \geq 0$ and $(c_j^T + \frac{b_j}{d_2} a_2^T) y \leq \frac{b_j}{d_2}$, $j = 1, \dots, k$.

e. Consider a company producing 2 products. Assume its production of product i is x_i . The profit of product i is $2x_1$ and $3x_2$, respectively. Whereas the cost of producing product i is $x_1 + 2$ and $2x_2 + 3$, respectively. The production constraints are $x_i \geq 0$ and $x_1 + x_2 \leq 10$. Assume the company wants to maximize its efficiency expressed as the ratio of total profit divided by total cost. Formulate the corresponding linear-fractional programming problem, and solve it using the above introduced equivalent linear programming problem.

EXERCISE 4.21 (Proof Proposition 4.3)

In this exercise we provide a proof for Proposition 4.3. Let $f(s)$ be differentiable on an open set $S \subset \mathbb{R}^n$ and $x \in S$. Then, f is differentiable in x in the direction u . Furthermore, the directional derivative of f in a in the direction u , denoted by $D_u f(x)$, equals

$$D_u f(x) = Df(x) * \frac{u}{\|u\|}.$$

Here, $\|u\|$ denotes the length of the vector $u = [u_1, \dots, u_n]^T$: $\sqrt{u_1^2 + \dots + u_n^2}$.

- Consider a fixed vector x in the interior of the domain of f and some fixed direction vector $u \in \mathbb{R}^n$. Show that there exist a $\delta > 0$ such that for all scalars $t \in (-\delta, \delta)$, $x + tu$ belongs to the interior of the domain of f .
- Show that the scalar function $F(t) := f(g(t))$, where $g(t) := x + tu$, is differentiable for $t \in (-\delta, \delta)$.
- Show that the derivative of the section function at point x equals $\lim_{t \downarrow 0} \frac{F(t) - F(0)}{t\|u\|}$.
- Show that $D_u f(x) = Df(x) * \frac{u}{\|u\|}$.

EXERCISE 4.22 (Exercise where KKT conditions don't reduce the set of potential solutions)

Consider the problem to minimize $f(x, y) = x^2 + y^2$ subject to the constraints $x \geq 0$; $y \geq 0$ and $xy = 0$.

- Show that all points where $xy = 0$ are obtained as potential candidates for a solution of the problem in Step 2 of Algorithm 4.20.
- Show, using Fritz-John's theorem, that all points (x, y) where $xy = 0$ qualify as a candidate solution.
- Solve the minimization problem.

EXERCISE 4.23 (*Optimization over the nonnegative orthant*)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous differentiable. Consider the problem to maximize $f(x)$ on the non-negative orthant, i.e., on the constraint set $\{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$.

Show that at a maximum location a next conditions apply:

1. $D_i f(a) \leq 0, i = 1, \dots, n$; 2. $a_i D_i f(a) = 0$; and 3. $a_i \geq 0, i = 1, \dots, n$.

EXERCISE 4.24 (*Barrier Interior-Point Method to solve numerically a constrained optimization problem*)

In this exercise we discuss a numerical approach, the so-called *Barrier Method*, to find the solution of a constrained optimization problem (under the assumption a solution exists).

Consider the problem to find the maximum of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ subject to $g(x) \leq b$.

- a. Let $I(x) : \mathbb{R} \rightarrow \mathbb{R}$ be the function that attains an arbitrarily small value $-\infty$ if $x > 0$ and a value 0 if $x \leq 0$. Assume there is just one constraint. Show that the constrained optimization problem has a solution iff. $f_I(x) := f(x) + I(g(x) - b)$ has a maximum (this result can be generalized simply to the case that there are m constraints).
- b. Let $t > 0$. Consider the function $\tilde{I}_t(x) = \frac{1}{t} \ln(-x)$ for $x < 0$. Plot $\tilde{I}_1(x)$, $\tilde{I}_{10}(x)$ and $\tilde{I}_{100}(x)$ in one graph. What can you say about $\tilde{I}_t(x)$ in relationship with $I(x)$.
- c. Consider the necessary conditions for the constrained maximization problem. Furthermore consider for a fixed $t > 0$, the first-order conditions for a maximum of the function $\tilde{f}_t(x) := f(x) + \frac{1}{t} \sum_{i=1}^m \ln(b_i - g_i(x))$ on the constrained set $S := \{x \mid g_i(x) < b_i, i = 1, \dots, m\}$. Assume that for all $t > 0$ this unconstrained optimization problem has a unique solution $x^*(t)$. Show that if $x^*(t)$ converges, for $t \rightarrow \infty$, this converged solution solves the KKT conditions of the constrained maximization problem. Hint: use $\lambda_i(t) := \frac{1}{t} \frac{1}{b_i - g_i(x^*(t))}$ to rewrite the first-order conditions in the unconstrained optimization problem.
- d. Consider the constrained maximization problem and the associated Lagrange function $L(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i(g_i(x) - b_i)$. Show that for all $\lambda \geq 0$, $L(x, \lambda) \geq f(x)$ and $\min_{\lambda \geq 0} L(x, \lambda) = f(x)$. Conclude from this that $f(x^*) = \max_{x \in S} \min_{\lambda \geq 0} L(x, \lambda)$ (i).
Show that with $\lambda_i := \lambda_i(t) := \frac{1}{t} \frac{1}{b_i - g_i(x^*(t))}$, $x^*(t)$ is a stationary point of $L(x, \lambda(t))$.
Assume $L(x, \lambda(t))$ has its unique maximum at $x^*(t)$ (ii).
Use (i) and (ii) to conclude that $f(x^*) \leq f(x^*(t)) + \frac{m}{t}$.
- e. Consider, for a fixed $t > 0$, the function $\tilde{f}_t(x) := f(x) + \frac{1}{t} \sum_{i=1}^m \ln(b_i - g_i(x))$ on the constrained set $S := \{x \mid g_i(x) < b_i, i = 1, \dots, m\}$.
Assume $\tilde{f}_t(x)$ has a unique maximum at $x^*(t)$ for all $t > 0$ and we want to approximate the maximum of our constrained optimization problem with an error of at most $\epsilon > 0$. Use item d to show that if we choose $t \geq \frac{m}{\epsilon}$ we achieve this goal.
- f. Consider for a fixed $t > 0$ the Newton-Raphson procedure (see Exercise 2.48) to approximate the maximum location $x^*(t)$ of $\tilde{f}_t(x)$.

$$x_{N+1} = x_N - H \tilde{f}_t^{-1}(x_N) D \tilde{f}_t^T(x_N), x_0 = \bar{x} \in S.$$

Show that

$$D\tilde{f}_t(x) = Df(x) - \frac{1}{t} \sum_{i=1}^m \frac{1}{b_i - g_i(x)} Dg_i(x) \text{ and}$$

$$H\tilde{f}_t(x) = Hf(x) - \frac{1}{t} \sum_{i=1}^m \left(\frac{1}{(b_i - g_i(x))^2} Dg_i^T(x) Dg_i(x) + \frac{1}{b_i - g_i(x)} Hg_i(x) \right).$$

- g. Since it is usually difficult to find for a large value of t the maximum location $x^*(t)$ of $\tilde{f}_t(x)$ one uses often a recursive procedure to find this location. One, often used, approach is to first approximate $x^*(t)$ for a smaller value of $\bar{t}$ (e.g. $\bar{t} = 1$) and next use this approximation $x^*(\bar{t})$ as an initialization point to approximate $x^*(t)$ for a larger value than $\bar{t}$. This process is then repeated until $t > \frac{m}{\epsilon}$, where ϵ is the maximal allowed error towards the sought-after maximum $f(x^*)$. Implement this method starting with $t = 1$ and doubling this value each iteration (using the Newton-Raphson procedure outlined in item f) to find the maximum of $f(x, y, z) = -x^2 - e^y - z^2$ on the constraint set $S = \{(x, y, z) \mid x^2 + y^2 \leq 16 \text{ and } x^4 + z^2 \leq 25\}$. Observe that with $t = 200$ we approximate the maximum of $-e^{-4}$ within a distance of 0.01.

EXERCISE 4.25 (Proof Theorem 4.35)

Consider the constrained maximization problem (4.1), where $f, g_i, i = 1, \dots, k, \in C^1$. Assume that the problem has a solution at $s = a$ and all constraints $g_i, i = 1, \dots, k_1$, are binding at this point while the remaining constraints are non-binding. Let

$$L(s, \lambda_0, \lambda_1, \dots, \lambda_k) := \lambda_0 f(s) - \lambda_1 (g_1(s) - b_1) - \dots - \lambda_k (g_k(s) - b_k)$$

denote the Lagrange function associated with this problem.

Below, you are asked to show - using Minkowski's¹⁶ separation theorem - that, for the Lagrangian function defined above, the necessary conditions stated in Theorem 4.35 apply.

Minkowski's separation theorem states that if $Z_i, i = 1, 2$, are two disjoint nonempty convex subsets of $\mathbb{R}^n$, then there exists a hyperplane $a^T x = b$ that separates both sets. That is,

$$a^T x \leq b, \text{ for all } x \in Z_1 \text{ and } a^T x \geq b, \text{ for all } x \in Z_2.$$

(see also Lemma 5.7, where you are asked to prove a special case of this result).

- Show that there does not exist a vector $d \in \mathbb{R}^n$ such that the inequalities $Df(a)d > 0$ and $Dg_i(a)d < 0, i = 1, \dots, k_1$, hold simultaneously.
- Define the sets

$$Z_1 := \{z \mid [-Df^T(a) \ Dg_1^T(a) \ \dots \ Dg_{k_1}^T(a)]^T z, x \in \mathbb{R}^n\}$$

¹⁶ Minkowski was a Polish mathematician who lived from 1864-1909

and

$$Z_2 := \{z \in \mathbb{R}^{k_1+1} \mid z_i < 0, i = 1, \dots, k_1 + 1\}.$$

Show that both sets are convex.

- c. Use Minkowski's separating hyperplane theorem to show that there exists a vector $a \geq 0$, with $a \neq 0$, such that $a^T z = 0$ for all $z \in Z_1$.
- d. Show that, for the Lagrangian function defined above, the necessary conditions stated in Theorem 4.35 apply.

EXERCISE 4.26 (*Proof Theorem 4.46*)

In this exercise we provide a proof for Theorem 4.46 using Farkas' lemma. Farkas' lemma states next result.

Let $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. Then exactly one of the next two assertions holds:

1. There exists an $x \in \mathbb{R}^m$ such that $Ax = b$ and $x \geq 0$.
 2. There exists a $y \in \mathbb{R}^n$ such that $A^T y \leq 0$ and $b^T y > 0$.
- a. Assume $B \in \mathbb{R}^{k \times n}$ and the dimension of the null space $\mathcal{N}(B)$ of matrix B is s . Show that $d \in \mathcal{N}(B)$ iff. there exists a $u \in \mathbb{R}^k$ such that $d = Hu$, where $H \in \mathbb{R}^{n \times s}$ is a full column rank matrix.
 - b. Assume the conditions of Theorem 4.46 are satisfied. Show that there exist matrices A_1 , B_1 and a vector c such that for all d satisfying $A_1 d \leq 0$ and $B_1 d = 0$ we have $c^T d \leq 0$.
 - c. Use item a and Farkas' lemma to show that under the conditions of Theorem 4.46 there exist a matrix $A^T := A_1 H$ and vector $b^T := c^T H$ such that $Ax = b$ and $x \geq 0$.
 - d. Use the fundamental theorem of linear algebra, that the null space $\mathcal{N}(B)$ of matrix B is orthogonal to the row space $\mathcal{R}(B^T)$, to obtain from item c the result that there exist $\lambda_i \geq 0$, $i = 1, \dots, k_1$ and μ_i , $i = 1, \dots, m$, such that $Df(a) = \sum_{i=1}^{k_1} \lambda_i Dg_i(a) + \sum_{i=1}^m \mu_i Dh_i(a)$.
 - e. Show that under the conditions of Theorem 4.46 the conditions (KKT) apply.

EXERCISE 4.27 (*Linear independence constraint qualification implies fulfilment Mangasarian-Fromovitz constraint qualification*)

Show that if the Linear independence constraint qualification is satisfied in Theorem 4.44, also the Mangasarian-Fromovitz constraint qualification is met. Hint: consider the equation $D\tilde{g}_1(a) = [-1 \ \dots \ -1 \ 0 \ \dots \ 0]^T$.

Problems with an Economic Content

EXERCISE 4.28 (consumer choice problem)

Consider a consumer who can buy basic consumption goods and luxury goods. The price of the consumption good is p_1 and that of the luxury good p_2 . The minimum amount of consumption goods he needs for living is m_1 . His total budget he can spend on the goods is b . Utility he obtains from buying an amount of x and y of the consumption and luxury goods is $U(x, y)$. Both products have the property that if one gets too much of it, utility from it declines.

- Formulate the constrained optimization problem for this consumer.
- Show the problem has no solution if the budget b is smaller than $p_1 m_1$.
- Assume the consumer is insatiable for either one of both goods. Show the problem attains its maximum either at $(m_1, \frac{b-p_1 m_1}{p_2})$, at $(\frac{b}{p_1}, 0)$ or at the point $(x, \frac{b-p_1 x}{p_2})$ where x solves the equation

$$\frac{p_1}{p_2} = \frac{U'_x}{U'_y}.$$

Explain under which conditions the solution $(m_1, \frac{b-p_1 m_1}{p_2})$ occurs.

- Next assume the consumer's utility from both products declines after a certain threshold $(\bar{x}, \bar{y})$ has occurred. That is: $U(x, y) \leq U(\bar{x}, \bar{y})$ for all positive values of x and y .

Consider a poor consumer which budget, $\bar{b}$, is insufficient to realize this optimum.

Calculate by how much his income should be raised to maximize his utility.

Another way to realize this maximum is that government lowers the prices of consumer goods. Indicate how one can calculate this new price which makes the consumer can realize this optimum utility.

EXERCISE 4.29 (optimal fertilization)

Consider a farmer who wants to grow two crops a year on his land of L acres. The growth curve of his crops is approximately given by the *logistic curves* $f_i(s) = \frac{\bar{f}_i}{1+e^{-(s-10)}}$, where $\bar{f}_i$ is the maximum yield that can be realized per acre for crop i , $i = 1, 2$. The farmer can use a different fertilizer for each crop to increase its yield. If he uses an amount of x_i per time unit of the fertilizer, the growth of the crop increases by a fraction of $\delta_i x_i$, $i = 1, 2$. Every year he can cultivate his land for an uninterrupted period of at most T time units. All prices the farmer has to pay for his fertilizer, p_i (per unit), and the prices he gets on the market for his crop, r_i (per unit), are fixed. Assume the farmer has a maximum budget b per year he wants to spend on fertilizer and he wants to maximize his income.

- Show that the optimization problem the farmer is confronted with can be formulated as the maximization w.r.t. the variables s_i and x_i , $i = 1, 2$, of the function:

$$\pi(s_1, s_2, x_1, x_2) := (1 + \delta_1 x_1) \frac{\bar{f}_1}{1+e^{-(s_1-10)}} L r_1 + (1 + \delta_2 x_2) \frac{\bar{f}_2}{1+e^{-(s_2-s_1-10)}} L r_2 - s_1 p_1 x_1 L - (s_2 - s_1) p_2 x_2 L$$

subject to $0 \leq s_1 \leq s_2 \leq T$; $s_1 p_1 x_1 L + (s_2 - s_1) p_2 x_2 L \leq b$ and $x_i \geq 0$.

- Let $\alpha_i := \delta_i \bar{f}_i r_i$, $i = 1, 2$. Show that if (s_1, s_2, x_1, x_2) solve the problem, there must exist

(s_1, s_3, x_1, x_2) and λ_i , $i = 1, \dots, 6$, satisfying next equations:

$$1. \text{ i) } \alpha_1 \frac{e^{-(s_1-10)}}{(1+e^{-(s_1-10)})^2} \left(\frac{1}{\delta_1} + x_1 \right) + \lambda_1 - \lambda_3 - (1 + \lambda_6) p_1 x_1 = 0;$$

$$\text{ii) } \alpha_2 \frac{e^{-(s_3-10)}}{(1+e^{-(s_3-10)})^2} \left(\frac{1}{\delta_2} + x_2 \right) + \lambda_2 - \lambda_3 - (1 + \lambda_6) p_2 x_2 = 0;$$

$$\text{iii) } ((1 + \lambda_6)s_1 - \lambda_4)(1 + e^{-(s_1-10)}) = \frac{\alpha_1}{p_1};$$

$$\text{iv) } ((1 + \lambda_6)s_3 - \lambda_5)(1 + e^{-(s_3-10)}) = \frac{\alpha_2}{p_2};$$

2. With $g_1 := -s_1$; $g_2 := -s_3$; $g_3 := s_1 + s_3 - T$; $g_4 := -x_1$; $g_5 := -x_2$; and $g_6 := s_1 p_1 x_1 L + s_3 p_2 x_2 L - b$, $\lambda_i g_i = 0$, $i = 1, \dots, 6$;

3. $\lambda_i \geq 0$ and $g_i \leq 0$, $i = 1, \dots, 6$.

Moreover, in that case $s_2 = s_3 + s_1$.

- c. Show from conditions 1.iii) and iv) that the farmer will always grow both crops.
- d. Show that if the farmer does not use fertilizer to increase the revenue for either one of his crops, to harvest optimally he should use the whole season T to grow his crops.
- e. Show that if a point $(\bar{s}_1, \bar{s}_3, \bar{x}_1, \bar{x}_2)$ in the interior of the constrained set and $\bar{\lambda}_i$, $i = 1, \dots, 6$, satisfy the necessary conditions in item b., the profit of the farmer is: $L(\frac{\bar{f}_1 r_1}{1+e^{-(\bar{s}_1-10)}} + \frac{\bar{f}_2 r_2}{1+e^{-(\bar{s}_3-10)}})$.
- f. Show that for the farmer it is always optimal to use either the whole growth season T to grow his crops and/or to spend his complete budget for fertilization. Hint: show that if an interior solution satisfies the necessary conditions one can always find a point in the constrained set where profit is higher.
- g. A numerical case study. Consider $p_1 = 1$, $p_2 = \frac{3}{2}$, $\delta_1 = 0.01$, $\delta_2 = 0.015$, $T = 30$, $b = 2.5 * 10^4$, $L = 10$, $\bar{f}_1 = 1500$, $\bar{f}_2 = 1300$, $r_1 = 1.0067$ and $r_2 = 1.0498$.

Use above items b-f and, e.g. Matlab, to verify next (approximate) results.

- i) Show that $(s_1, s_3, x_1, x_2) = (15, 13, 11.16, 107.19)$ is an interior solution that satisfies the necessary conditions.
- ii) Show that if the budget restriction is not binding next candidate solutions result:
 $(s_1, s_3, x_1, x_2) = (17, 13, 0, 104.8)$ with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0, 0, 1.37, 1.91, 0, 0)$.
 $(s_1, s_3, x_1, x_2) = (15, 15, 1.074, 0)$ with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0, 0, 9.073, 0, 1.444, 0)$.
- iii) Show that if not the whole growth season is used next (approximate) candidate solution results:
 $(s_1, s_3, x_1, x_2) = (14.9764, 12.9527, 11.4345, 119.8507)$ with
 $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0, 0, 0, 0, 0, 0.0014)$.
- iv) Show that if as well the whole budget for fertilization is used as crops are grown for the whole growth season $(s_1, s_3, x_1, x_2) = (12.9171, 17.0829, 193.5419, 0)$ and
 $(s_1, s_3, x_1, x_2) = (17.084, 12.916, 0, 129.0389)$ result as candidate solutions.
- v) Show that in the optimal situation the farmer should spend all his budget for fertilization on the first crop for approximately 13 time units and next use the rest of the growth season to grow his second crop.

EXERCISE 4.30 (green farming)

Consider a farmer who grows an amount x of a product using traditional farming and an amount y using a more environmentally friendly way. He can grow at most a total amount c of the product. The price he can get on the market for his product using traditional farming is $p_1(x, y)$ and using the environmentally friendly way is $p_2(x, y)$, where $p_1(x, y) \leq p_2(x, y)$. The production cost are $C_1(x)$ and $C_2(y)$, respectively, where $C_1(z) \leq C_2(z)$. All functions are assumed to be continuous.

- a. Assume the farmer wants to maximize his profit. Formulate the optimization problem and show the problem has a solution.
- b. Formulate the KKT conditions.
- c. Consider the case $p_i(x, y) = \alpha_i - (x + y)$ and $C_i(z) = \beta_i z + \gamma_i$, $i = 1, 2$, where the constants α_i, β_i and γ_i are all nonnegative and satisfy $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$, $\gamma_1 \leq \gamma_2$ and $\beta_1 - \alpha_1 \neq \beta_2 - \alpha_2$. Show that the problem has no interior solution.
- d. Show that under the assumption that $(\frac{\alpha_i - \beta_i}{2})^2 \neq \gamma_1 - \gamma_2$, $i = 1, 2$, the constraint qualification is met.
- e. Show from the KKT conditions that, under the parametric assumptions stated in items c and d, if the farmer decides to produce he will always specialize in the production of either one of both products. Show he will specialize in the environmentally friendly way iff. $\alpha_2 - \alpha_1 > \beta_2 - \beta_1$.
- f. Assume the farmer grows the product using traditional farming and the government wants the farmer to grow it in the environmentally way. Indicate how the government might achieve this goal.

CHAPTER 5

Constrained Optimization: The Convex and Linear Case

In the previous sections, we derived the necessary KKT conditions for the existence of a solution to the constrained optimization problem (4.8). In this section, we focus on the case where the objective function is concave and the constraint functions are convex, since many optimization problems can be reformulated within this framework. These are referred to as *convex programming problems*, which possess several desirable properties. The most important of these properties will be discussed below. In particular, we will also consider the subclass of problems in which all functions are linear.

Because convex optimization problems play such a central role in practical applications, considerable effort has been devoted to solving them efficiently from a computational perspective. We will largely omit this discussion, as it falls outside the scope of this book. Readers interested in computational methods are referred to the extensive literature on convex and linear programming; see, for example, [3], [5], [17], [21], [22], and [23].

In the study of convex (and concave) functions, convex sets play a fundamental role. Therefore, we begin this section with an elementary discussion of convex sets and their relationship to the optimization of convex functions under convex constraints.

5.1 Convex sets and their relationship with convex functions

In the study of convex programming problems, the objective function (in the case of minimization) and the constraint functions are assumed to be convex, which requires that they be defined on a convex set. As we shall see, this assumption implies that the feasible set for such problems is itself convex. Therefore, we begin this section by recalling the definition of a convex set and examining some of its key properties in more detail.

Recall from Chapter 2 that a set S is called convex if, for any two points in S , the line segment connecting them also lies entirely within S . That is:

DEFINITION 5.1 Let $S \subset \mathbb{R}^n$. The set S is called *convex* if, for any $s_1, s_2 \in S$, the point

$$s = \alpha s_1 + (1 - \alpha)s_2$$

also belongs to S for every $\alpha \in [0, 1]$.

It is easy to verify the following property: if two sets in $\mathbb{R}^n$ are convex, then their intersection is also convex, provided it is nonempty.

LEMMA 5.2 Let $S, T \subset \mathbb{R}^n$ be convex. Then $S \cap T$ is either empty or convex.

EXERCISE 5.1 Prove above Lemma 5.2.

Note that this property does not generally hold for the union of convex sets, as illustrated in the next example.

EXAMPLE 5.3 Consider the sets $S = \{1\}$ and $T = \{2\}$ in $\mathbb{R}$. Then $S \cup T = \{1, 2\}$ is not a convex set in $\mathbb{R}$.

A property frequently used in convex analysis is that if a constraint set is defined by a convex function over a convex set, then the resulting set is either empty or convex. Lemma 5.4 below provides the formal statement, and you are asked to prove it in Exercise 5.2.

LEMMA 5.4 Let $S \subset \mathbb{R}^n$ be convex, $g : S \rightarrow \mathbb{R}$ be a convex function and $b \in \mathbb{R}$. Consider $G := S \cap \{x \in \mathbb{R}^n \mid g(x) \leq b\}$. Then G is either empty or convex.

EXERCISE 5.2 Prove above Lemma 5.4.

From Lemmas 5.4 and 5.2, we can conclude in particular that if, in the standard formulation of a constrained optimization problem, all functions g_i are convex and defined on convex sets C_i , $i = 1, \dots, m$, then the constraint set is either empty or convex. Put differently, in convex optimization problems, the feasible set is convex.

PROPOSITION 5.5 Let $C_i \subset \mathbb{R}^n$ be convex, $g_i : C_i \rightarrow \mathbb{R}$ be convex functions and $b_i \in \mathbb{R}$, $i = 1, \dots, m$. Then the constraint set $G := \{x \in \mathbb{R}^n \mid g_i(x) \leq b_i, i = 1, \dots, m\}$ is either empty or convex.

Recall from Chapter 3 that if a concave function on $\mathbb{R}^n$ has an extremum, that point corresponds to a global maximum of the function. Using Proposition 5.5 above, this result can be easily extended to convex programming problems. Assuming the constraint set of the problem is nonempty, it follows directly from the definitions that any solution of a convex programming problem provides a global maximum.

Moreover, if the objective function is strictly concave¹⁷ the maximum is attained at a unique point, assuming a maximum exists. In Exercise 5.21, you are asked to prove these statements. Theorem 5.6 summarizes these results.

THEOREM 5.6 *Consider, without any differentiability assumptions, the constraint optimization problem (4.8) on a convex set C where f is concave and g_i , $i = 1, \dots, m$, are convex.*

Assume this convex programming problem has a solution.

Then, if f attains a maximum at $\bar{x}$, $f(\bar{x})$ is a global maximum. Moreover, if f is strictly concave, there is no other point where this maximum is attained.

An important fundamental property established by Minkowski [15] is that a convex set and any point outside it can always be separated by a (hyper)plane. This result is known in the literature as the *Separating Hyperplane Theorem* and is formalized in Lemma 5.7 below. You are asked to prove this lemma in Exercise 5.14, item g.

LEMMA 5.7 (*Separating Hyperplane Theorem*)

Let $S \subset \mathbb{R}^n$ be a convex set. Let $y \notin S$. Then there exists a plane $H : \{x \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = b\}$ that separates y and S . That is, there is a plane H such that y is located to one side of H and S to the other side. Or, more formal, there exists a vector $a \in \mathbb{R}^n$, $a \neq 0$, such that $a^T x \geq a^T y$ for all $x \in S$. Moreover, if S is closed, this inequality is strict.

The lemma above is, for example, used to prove Lemma 5.8, which provides a fundamental result from which many properties of convex functions can be derived. In general, support and separation properties play a central role in the development of linear programming, convex analysis, and optimization theory. A survey of support and separation properties of convex sets in n -dimensional Euclidean space can be found in, for example, [25]. Lemma 5.8 was established by Fan, Glicksberg, and Hoffman in [8].

In Exercise 5.15, you are asked to provide a proof, which follows the approach given by Takayama in [28][Theorem 1.B.2] (see also [7][Lemma 6.3]).

¹⁷ f is strictly concave on S if for all $x, y \in S$, $x \neq y$, and $\lambda \in (0, 1)$, $f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$.

LEMMA 5.8 (*Fan, Glicksberg and Hoffman*)

Let $X \subset \mathbb{R}^n$ be convex and $f_i : X \rightarrow \mathbb{R}$, $i = 1, \dots, m$, be concave.

Consider the set

$$S := \{x \mid f_i(x) > 0, i = 1, \dots, m\}.$$

Then, if $S = \emptyset$ there exists a $\lambda_i \geq 0$, $\lambda \neq 0$, such that

$$\sum_{i=1}^m \lambda_i f_i(x) \leq 0, \forall x \in X.$$

5.2 Convex Programming and Saddle-Point Equilibria

As noted at the beginning of this chapter, we present here only some of the main results for convex programming problems. We first discuss results that make no differentiability assumptions on the functions involved and relate these results to the existence of saddle points for the associated Lagrange function. We then introduce differentiability assumptions and derive the corresponding KKT conditions.

For clarity, the proofs of Theorems 5.9, 5.10, and 5.13 have been moved to the advanced exercises 5.19, 5.20, and 5.23, respectively. Many additional results can be found in the literature, for example in [5], [21], [22], [23], and [28].

In the results presented below, the Lagrange function

$$L(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i (g_i(x) - b_i) =: f(x) - \lambda^T (g(x) - b)$$

again plays an important role, where $\lambda \in \Lambda := \{\lambda \in \mathbb{R}^m \mid \lambda_i \geq 0, i = 1, \dots, m\}$.

Our first result states that if the Lagrange function has a saddle point $(\bar{x}, \bar{\lambda})$, then the constraint optimization problem (4.8) attains its global maximum at $\bar{x}$.

THEOREM 5.9 *Let f, g_i be defined on some subset X of $\mathbb{R}^n$.*

Then, if there exists a point $(\bar{x}, \bar{\lambda}) \in X \times \Lambda$ such that

$$L(x, \bar{\lambda}) \leq L(\bar{x}, \bar{\lambda}) \leq L(\bar{x}, \lambda), \forall x \in X, \forall \lambda \in \Lambda,$$

1. *the constraint optimization problem (4.8) attains its global maximum at $\bar{x}$.*
2. *$\bar{\lambda}^T (g(\bar{x}) - b) = 0$, i.e., $\bar{\lambda}_i = 0$ and/or $g_i(\bar{x}) = b_i$, $i = 1, \dots, m$.*

The converse of Theorem 5.9 holds when additional assumptions are imposed on the objective function, the constraint functions, and the domain X .

THEOREM 5.10 *Let f and g_i be functions defined on a convex subset X of $\mathbb{R}^n$ with f concave and g_i convex. Moreover, assume that g_i satisfy Slater's condition*

$$\exists \tilde{x} \in X \text{ such that } g_i(\tilde{x}) - b_i < 0, \quad i = 1, \dots, m. \quad (5.1)$$

Then, if $f(x)$ has a maximum at $\bar{x}$ subject to the constraints $g(x) \leq b$, $x \in X$, there exists a $\bar{\lambda} \in \Lambda$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the Lagrange function $L(x, \lambda)$, i.e.

$$L(x, \bar{\lambda}) \leq L(\bar{x}, \bar{\lambda}) \leq L(\bar{x}, \lambda), \quad \forall x \in X, \forall \lambda \in \Lambda.$$

The proof of Theorem 5.10 (see Exercise 5.20) can be obtained by using Lemma 5.8. From Theorems 5.9 and 5.10, we immediately obtain the following corollary if differentiability assumptions are once again imposed on the functions involved.

COROLLARY 5.11 *Let f and g_i be differentiable functions defined on an open convex subset X of $\mathbb{R}^n$ with f concave and g_i convex. Moreover, assume that g_i satisfy Slater's condition (5.1).*

Then the constraint optimization problem (4.8) has a maximum if and only if the KKT conditions (1)-(3) apply. Moreover, if a maximum exists it is a global maximum.

EXAMPLE 5.12 Consider the maximization of $f(x, y) = \ln(1+x) - y^2$ subject to the constraints $x \geq 0$ and $x^2 + y^2 \leq 4$.

Define $g_1(x, y) := -x$ and $g_2(x, y) := x^2 + y^2 - 4$. Then

$$Df(x, y) = \left[\frac{1}{1+x}, -2y \right], \quad Dg_1 = [-1, 0] \text{ and } Dg_2(x, y) = [2x, 2y].$$

All partial derivatives are continuous on the feasible set, and hence f , g_1 and g_2 are differentiable on this set.

Moreover,

$$Hf(x, y) = \begin{bmatrix} -\frac{1}{(1+x)^2} & 0 \\ 0 & -2 \end{bmatrix}, \quad Hg_1(x, y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Hg_2(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Thus, Hf is negative semi-definite on its domain while both Hg_i , $i = 1, 2$ are positive semi-definite. It follows that f is concave and g_1 and g_2 are convex on the feasible set. Finally, observe that at, for example, $(1, 1)$, both constraints satisfy $g_i(1, 1) < 0$, $i = 1, 2$.

Hence, all conditions of Corollary 5.11 are satisfied. According to this corollary, all solutions of the problem are obtained from the with this problem associated Kuhn-Tucker conditions.

Consider the corresponding Lagrange function

$$L(x, y, \lambda_1, \lambda_2) := \ln(1+x) - y^2 + \lambda_1 x - \lambda_2(x^2 + y^2 - 4).$$

The associated Kuhn-Tucker conditions are:

1. $DL = [\frac{1}{1+x} + \lambda_1 - 2\lambda_2 x, -2y - 2y\lambda_2] = [0 \ 0]$.
2. i) $\lambda_1 x = 0$ and ii) $\lambda_2(x^2 + y^2 - 4) = 0$.
3. i) $x \geq 0$, ii) $x^2 + y^2 \leq 4$ and iii) $\lambda_i \geq 0, i = 1, 2$.

From $D_2 L = 0$ and item 3.iii), it follows that $y = 0$. Substituting this into the above conditions and subsequently evaluating all combinations of binding constraints, we obtain:

1. Suppose first that no constraint is binding. Then $\lambda_1 = \lambda_2 = 0$, but in this case the equation $D_1 L = 0$ has no solution.
2. Next, consider the case in which exactly one constraint is binding.
 Case 1: $x = 0$ and $\lambda_2 = 0$. From $D_1 L = 0$ we obtain $\lambda_1 = -1$, which violates item 3.iii).
 Case 2: $x^2 + y^2 = 4$ and $\lambda_1 = 0$. Since $y = 0$, we obtain $x = 2$. Substitution into $D_1 L = 0$ yields $\lambda_2 = \frac{1}{12}$.
3. The case in which both constraints are binding simultaneously is not possible since $y = 0$.
 Consequently, by Corollary 5.11, the problem has a global minimum, which is attained at $(x, y) = (2, 0)$.

EXERCISE 5.3 Consider the problem to minimize $f(x, y) = (x+1)^2 + (y+1)^2$, subject to the constraints $(x-1)^2 \leq 4$, and $y \geq 0$.

- a. Is the constraint set compact?
- b. Solve the problem using Corollary 5.11.
- c. Show that the Lagrangian $L(x, y, \lambda_1, \lambda_2) := -(x+1)^2 - (y+1)^2 - \lambda_1((x-1)^2 - 4) + \lambda_2 y$ has a saddle-point equilibrium at $((x, y), (\lambda_1, \lambda_2)) = ((-1, 0), (0, 2))$.

When the constraints g_i are linear, the result stated in Corollary 5.11 can be further simplified, as Slater's condition is no longer required (see Exercise 5.23). This is formalized in Theorem 5.13.

THEOREM 5.13 Let f be a concave differentiable function defined on an open convex subset X of $\mathbb{R}^n$. Assume that all functions $g_i(x)$ are linear.

Then the constraint optimization problem (4.8) has a maximum at $\bar{x}$ if and only if the KKT conditions (1)-(3) apply. Or, equivalently, there exists a $\bar{\lambda} \in \Lambda$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the Lagrange function $L(x, \lambda)$.

We emphasize that in both Corollary 5.11 and Theorem 5.13, there is no need to check whether the solutions satisfying the KKT conditions (1)–(3) are actually optimal. Any of these solutions automatically yields the (global) maximum of the constrained optimization problem (4.8).

EXERCISE 5.4 Use Theorem 5.13 to solve the next constrained optimization problems.

- Maximize $U(x, y) = x + \ln(1 + y)$, subject to the constraints $x + 2y \geq 4$, $x \geq 0$ and $y \geq 0$.
- Minimize $\tilde{U}(x, y) = x - \ln(1 + y)$, subject to the constraints $x + 2y \geq 4$, $x \geq 0$ and $0 \leq y \leq 10$.
- Show that the Lagrangian $L(x, y, \lambda) := -x + \ln(1 + y) - \lambda_1(4 - x - 2y) + \lambda_2x + \lambda_3y - \lambda_4(y - 10)$, associated with the problem considered in item b., has a saddle-point at $((x, y), (\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = ((0, 10), (0, 1, 0, \frac{1}{11}))$.

5.3 Linear Programming and its Dual Problem

In this section, we briefly highlight a special case from the previous section, namely when both the objective function and the constraint functions are linear. As noted earlier, there is a vast literature on this topic. Readers seeking more details, particularly those interested in implementing a linear program, are referred to these sources.

Below we consider next standard formulation of the linear programming problem.

Standard Problem Formulation Linear Programming (LP)

Let $c, a_i \in \mathbb{R}^n / \{0\}$ and $b \in \mathbb{R}^m$. Find the maximum w.r.t. $x_i \geq 0, i = 1, \dots, n$, of

$$c^T x, \text{ subject to } a_j^T x \leq b_j, j = 1, \dots, m. \quad (5.2)$$

By introducing matrix A as the matrix which i^{th} row equals a_i^T , above LP problem can be reformulated in shorthand as:

find the maximum of $c^T x$ subject to the constraint set defined by $Ax \leq b$ and $x \geq 0$.

Notice that each constraint in LP problem above describes a half-space (i.e., in $\mathbb{R}^2$, all points on one side of a line; in $\mathbb{R}^3$, all points on one side of a plane; and so on). Geometrically, the feasible set is therefore the intersection of these half-spaces. In the literature such a set is called a *polyhedron*.

If a point c on the boundary of a polyhedron satisfies n linearly independent binding constraints, c is called a *corner point* or *extreme point* of the polyhedron. In an LP problem, the n constraints $x_i = 0, i = 1, \dots, n$, are always linearly independent, which ensures that the feasible set of an LP problem always has at least one corner point.

These corner points play a key role in solving an LP problem. Since the objective function is linear, it can be shown that if the LP problem has a solution, this solution is attained at least at one of its corner points. Furthermore, because any polyhedron is a convex set (see, e.g., Lemma 5.2 and Lemma 5.4 again), the linearity of the objective function (which in particular means that it is concave) guarantees that any maximum found is a global maximum.

Corollary 5.15, below, summarizes key information regarding the existence and location of solution(s).

The Lagrange function (5.3) associated with this LP problem plays a central role in obtaining these results.

$$L(x, \lambda, \mu) := c^T x - \lambda^T (Ax - b) + \mu^T x, \text{ where } \lambda \geq 0 \text{ and } \mu \geq 0. \quad (5.3)$$

From Theorem 5.13 we immediately obtain next result for the LP problem.

THEOREM 5.14 *The LP problem (5.2) has a maximum at x if and only if there exist λ , μ and x satisfying next three conditions:*

- i. $c^T - \lambda^T A + \mu^T = 0$;
- ii. $\lambda_i(a_i^T x - b_i) = 0$, $i = 1, \dots, m$, and $\mu_i x_i = 0$, $i = 1, \dots, n$;
- iii. $\lambda \geq 0$; $\mu \geq 0$; $Ax \leq b$ and $x \geq 0$.

An important property of linear objective functions is that, when evaluated along a line in any direction, the function values either strictly increase, strictly decrease, or remain constant along that line (see Exercise 5.25). From this property and Theorem 5.14, the results summarized in Corollary 5.15 follow. Readers are asked to work out the detailed proofs in Exercises 5.26 and 5.27.

COROLLARY 5.15 *Consider the LP problem (5.2). Then next properties hold.*

1. *The LP problem has either no, a unique or an infinite number of solutions.*
2. *If the LP problem has a solution, it is attained at the border of the constraint set.*
3. *If the LP problem is solved at $\bar{x}$ and $\bar{y}$ then it is also solved at any point on the line segment between $\bar{x}$ and $\bar{y}$. More general, if the LP problem has a solution at n different points, then the LP problem is also solved at any point in the convex hull¹⁸ of these points.*
4. *If the LP problem has a solution, then there is at least one corner point where this solution is attained.*

The observations above are very helpful in designing numerical algorithms for efficiently solving LP problems. To gain some intuition about how these conditions are used in practice, consider the case of two variables, i.e., $n = 2$.

In this case, the level curves of the objective function are lines. Recall that each constraint applies to all points on one side of a line. This implies that, if the problem has a solution, it will occur at the intersection of two binding constraints - that is, at the intersection of two lines that define the boundary of the feasible set. In the terminology introduced earlier, this means that the solution occurs at a corner point of the feasible set.

Figure 5.1 illustrates this situation.

¹⁸ The *convex hull* of the points x_i , $i = 1, \dots, n$, is the set of all convex combinations of these points, i.e. $\{x | x = \sum_{i=1}^n \alpha_i x_i, \text{ with } \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0\}$. Geometrically it is the smallest convex subset containing these n points.

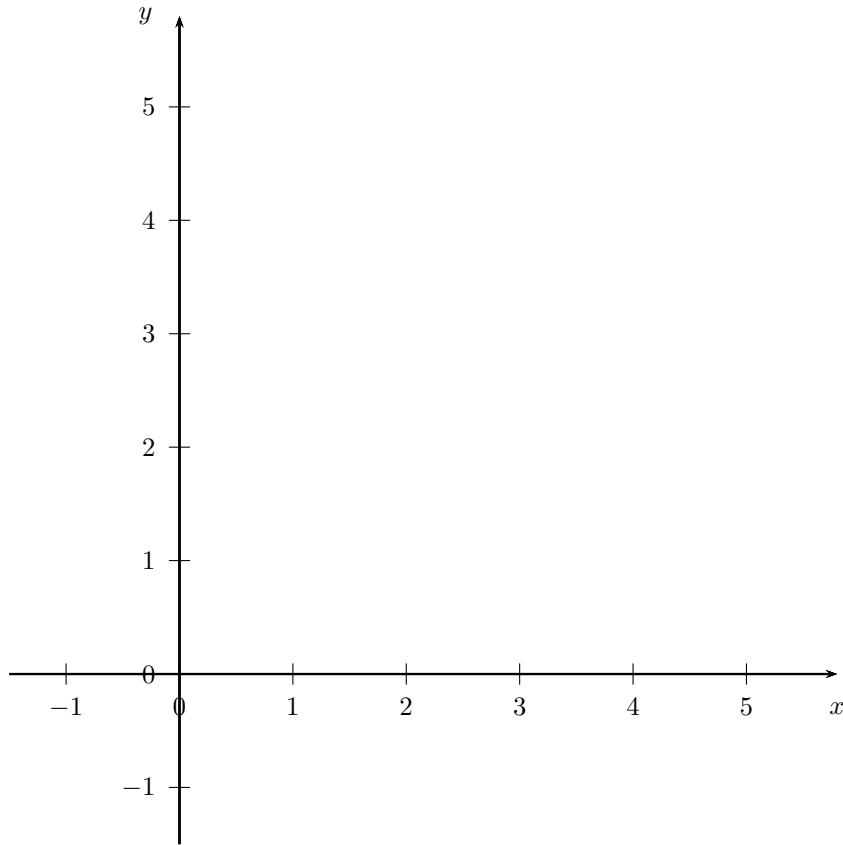


FIGURE 5.1 – In grey: feasible set LP problem. The dotted lines represent level curves. The dots indicate the corner points. At at least one of these dots a maximum is attained if the problem has a solution (here: $(0, 4)$).

Now assume that no more than two constraints are binding at any point in the feasible set (a condition that can be easily imposed and checked as a rank condition on a matrix associated with the problem). Then, conditions ii. and iii. of Theorem 5.14 imply that all λ and μ , except for two, must be zero.

By evaluating the objective function at these corner points, one can identify the point(s) where the largest objective value is attained. By then verifying whether, at this (these) extremal corner point(s), with the corresponding zero Lagrange variables, the remaining conditions of the theorem are satisfied, one can confirm whether an optimum exists and whether its location is unique. If the constraint set is compact, this verification step can even be skipped, since by Weierstrass's theorem, the LP problem is guaranteed to have a solution. However, note that the maximum value may be attained at infinitely many points.

This is the basic idea behind solving linear programming problems using the so-called *simplex method*.

We will not go into the details of this numerical method here, but the procedure for solving a simple LP problem is summarized in Algorithm 5.16.

ALGORITHM 5.16 To find the solution(s) of the LP problem (5.2) proceed as follows:

Step 1. Find all corner points, c_i , of the constraint set. This can be achieved by calculating the intersection point of any combination of n binding constraints out of the $n + m$ constraints and next verify whether this point belongs to the constraint set.

Step 2. Evaluate the objective function at all corner points c_i . Let m be the maximum of all these numbers. If m is uniquely attained at the corner point c_j let $S := \{c_j\}$. If m is attained at the corner points $c_j, \dots, c_k$, let $S := \text{convex hull} \{c_j, \dots, c_k\}$.

Step 3. Verify whether the constraint set is bounded.

If this set is bounded the solution of the LP problem is m and it is attained at S .

If this set is unbounded verify whether the conditions mentioned in Theorem 5.14 apply at S . If these conditions apply the solution of the LP problem is m and it is attained at S . Otherwise the LP problem has no solution.

Another way to demonstrate that the LP problem has no solution in Step 3 of the above algorithm is, of course, to find a point in the feasible set where the objective function attains a value larger than m .

EXERCISE 5.5 Rewrite next linear programming problems into the standard form and next solve them.

- Maximize $f(x, y) = x + 3y$, subject to the constraints $x \leq 4$, $y \leq 4$, $2x + y \leq 9$, $x + 2y \leq 9$, $x + y \leq 6$, $x \geq 0$ and $y \geq 0$.
- Minimize $f(x, y) = 6x + y$, subject to the constraints $x + y \geq 1$, $x + 100y \geq 100$, $100x + y \geq 100$, $x \geq 0$ and $y \geq 0$.
- Maximize $U(x, y) = x + y$, subject to the constraints $x + 3y \leq 10$, $3x + y \leq 10$, $2x + 2y \leq 8$, $x \geq 0$ and $y \geq 0$. Solve the problem first by using Theorem 5.14. Next, solve the problem using Corollary 5.15.
- Maximize $f(x, y) = x - y$ subject to the constraints $x + 3y \leq 40$, $3x + y \leq 16$, $x + y \leq 15$, $x \geq 1$ and $y \geq 10$.

Associated with the linear programming problem 5.2 is its so-called *dual problem*. As we will see shortly, this is also a linear programming problem. Moreover, the solutions of the primal and dual problems are directly related. In some cases, the dual problem may have a smaller dimension than the original LP problem, making it more efficient to solve.

Recall from Theorem 5.13 that the LP problem 5.2 has a solution if and only if there exists $(\bar{\lambda}, \bar{\mu}) \in \Lambda$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ forms a saddle point of the Lagrange function (5.3). That is, if

$L(x, \lambda, \mu) = c^T x - \lambda^T (Ax - b) + \mu^T x$, with $(\lambda, \mu) \in \Lambda$, they satisfy:

$$L(x, \bar{\lambda}, \bar{\mu}) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(\bar{x}, \lambda, \mu). \quad (5.4)$$

Next, observe that

$$\begin{aligned} L(x, \lambda, \mu) &= c^T x - \lambda^T (Ax - b) + \mu^T x \\ &= x^T c - (x^T A^T - b^T) \lambda + x^T \mu \\ &= b^T \lambda - x^T (A^T \lambda - c - \mu). \end{aligned}$$

With

$$\begin{aligned} \Lambda_D(\lambda, \mu, x) &:= -\Lambda(x, \lambda, \mu) \\ &= -[b^T \ 0] \begin{bmatrix} \lambda \\ \mu \end{bmatrix} - x^T ([-A^T \ I] \begin{bmatrix} \lambda \\ \mu \end{bmatrix} + c) \end{aligned}$$

we now immediately obtain from (5.4) that $(\bar{\lambda}, \bar{\mu}, \bar{x})$ satisfies

$$L_D(\lambda, \mu, \bar{x}) \leq L_D(\bar{\lambda}, \bar{\mu}, \bar{x}) \leq L_D(\bar{\lambda}, \bar{\mu}, x).$$

Thus, by Theorem 5.13, the problem of maximizing (w.r.t. $\lambda \geq 0$ and $\mu \geq 0$) of

$$-b^T \lambda, \text{ subject to } -A^T \lambda + \mu + c \leq 0, \quad (5.5)$$

has a solution $(\bar{\lambda}, \bar{\mu})$.

Furthermore, from the KKT conditions derived in the first part of Theorem 5.13, we conclude that

$$\bar{x}^T (-A^T \bar{\lambda} + \bar{\mu} + c) = 0.$$

This then implies that

$$\begin{aligned} c^T \bar{x} &= c^T \bar{x} - \bar{\lambda}^T (A\bar{x} - b) + \bar{\mu}^T \bar{x} \\ &= b^T \bar{\lambda} - \bar{x}^T (-A^T \bar{\lambda} + \bar{\mu} + c) = b^T \bar{\lambda}. \end{aligned}$$

It is straightforward to verify that the maximization problem (5.5) has a solution if and only if the problem of maximizing (w.r.t. $\lambda \geq 0$) of

$$-b^T \lambda, \text{ subject to } -A^T \lambda + c \leq 0,$$

has the solution $\bar{\lambda}$. Equivalently, this is the same as saying that the following minimization problem has a solution:

Dual Problem of Standard LP Problem

Find the minimum w.r.t. $\lambda \geq 0$ of

$$b^T \lambda, \text{ subject to } A^T \lambda - c \geq 0. \quad (5.6)$$

Problem (5.6) is called the *dual linear programming problem* of (5.2).

So, we showed next result.

THEOREM 5.17 *The linear programming problem (5.2) has a solution at $\bar{x}$, with associated co-state variables $\bar{\lambda} \geq 0$, if and only if the dual linear programming problem (5.6) has a solution at $\bar{\lambda} \geq 0$, with associated co-state variables $\bar{x} \geq 0$. Moreover, if the problems have a solution the optimal values coincide, i.e., $b^T \bar{\lambda} = c^T \bar{x}$.*

REMARK 5.18 An alternative way to obtain the above result is to consider the so-called Lagrange dual function. To motivate this approach, we revisit the general problem formulation (4.8). The solution of this problem coincides with that of the unconstrained optimization problem of maximizing

$$F(x) := f(x) - \sum_{i=1}^n G(g_i(x)), \quad x \in C, \quad (5.7)$$

where $G(g_i(x)) = 0$ if $g_i(x) - b_i \leq 0$ and $G(g_i(x)) = \infty$ if $g_i(x) - b_i > 0$.

Now consider the Lagrange function

$$L(x, \lambda) := f(x) - \sum_{i=1}^n \lambda_i (g_i(x) - b_i), \quad x \in C.$$

For any fixed $x \in C$ the value of $\inf_{\lambda_i \geq 0} L(x, \lambda)$ coincides with $F(x)$. Indeed, if $g_i(x) - b_i \leq 0$, the infimum is attained at $\lambda_i = 0$, otherwise, the infimum is achieved by letting λ_i grow arbitrarily large. Consequently, the optimum of our problem (4.8) can be found by solving

$$\max_{x \in C} \inf_{\lambda_i \geq 0} L(x, \lambda).$$

Next, we consider the problem obtained by reversing the order of maximization and minimization in the formulation above. This leads to the study of the associated *Lagrange dual function*, defined by

$$L_D(\lambda) := \sup_{x \in C} L(x, \lambda).$$

If Problem (4.8) has a solution x^* with $f(x^*) = p$, it can be shown that for every λ , $L_D(\lambda) \geq p$ (see Exercise 5.29). In particular,

$$d := \inf_{\lambda \in \Lambda} L_D(\lambda) \geq p.$$

In the literature, the difference $d - p$ is referred to as the *duality gap*.

For convex optimization problems, it can be shown - assuming, for example, that Slater's condition holds - that this duality gap is zero. Moreover, $L_D(\lambda)$ is a convex function (see Exercise 5.30). As a result, from a numerical point perspective it is typically not difficult to compute d , that is, to determine an upper bound for the optimal value of the optimization problem (4.8). This property provides strong motivation for using this approach, especially when dealing with complex optimization problems.

EXAMPLE 5.19 Consider the linear programming problem of maximizing

$$f(x_1, x_2) = 2x_1 + 3x_2 \text{ subject to } g_1(x_1, x_2) := x_1 + x_2 \leq 5, \quad g_2(x_1, x_2) := 2x_1 + x_2 \leq 8,$$

and $x_i \geq 0$, $i = 1, 2$.

Observe that the feasible set is a polytope with four "corner-points": $(0, 0)$; $(4, 0)$; $(0, 5)$ and $(3, 2)$. Since this constraint set is compact Weierstrass's theorem guarantees the existence of a solution. Evaluating f at these points corner-points shows that the maximum value, equal to 15, is attained at $(0, 5)$.

Let

$$c := \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \quad A := \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

The corresponding dual problem is then given by

$$\text{Minimize } f_D(\lambda_1, \lambda_2) := 5\lambda_1 + 8\lambda_2$$

subject to

$$g_{D1} := \lambda_1 + 2\lambda_2 \geq 2 \text{ and } g_{D2} := \lambda_1 + \lambda_2 \geq 3.$$

The feasible set of the dual problem has two "corner-points", $(0, 3)$ and $(3, 0)$. Evaluating f at these points shows that the minimum value, equal to 15, is attained at $(3, 0)$.

The following example illustrates that, particularly when the number of variables exceeds the number of constraints, it may be easier to solve the corresponding dual problem of the linear program.

EXAMPLE 5.20 Consider the linear programming problem of maximizing

$$f(x_1, x_2, x_3) = 10x_1 + 6x_2 + 8x_3$$

subject to

$$g_1(x_1, x_2, x_3) := x_1 + x_2 + 2x_3 - 2 \leq 0, \quad g_2(x_1, x_2, x_3) := 5x_1 + 3x_2 + 2x_3 - 1 \leq 0,$$

and $x_i \geq 0$, $i = 1, 2, 3$.

Let

$$c := \begin{bmatrix} 10 \\ 6 \\ 8 \end{bmatrix}; \quad A := \begin{bmatrix} 1 & 1 & 2 \\ 5 & 3 & 2 \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The corresponding dual problem is then

$$\text{Minimize } f_D(\lambda_1, \lambda_2) := 2\lambda_1 + \lambda_2$$

subject to

$$g_{D1} := \lambda_1 + 5\lambda_2 \geq 10 \quad g_{D2} := \lambda_1 + 3\lambda_2 \geq 6 \quad \text{and} \quad 2\lambda_1 + 2\lambda_2 \geq 8.$$

This dual problem can be solved by evaluating f_D at the three "corner-points" $(0, 4)$; $(10, 0)$ and $(\frac{5}{2}, \frac{3}{2})$. The minimum value, equal to 4, is attained at $(0, 4)$.

To determine the solution of the original linear program, this information can be used to simplify the associated KKT conditions. These conditions are given by

$$\begin{aligned} i) \quad & 10 - \lambda_1 - 5\lambda_2 + \mu_1 = 0; \quad ii) \quad 6 - \lambda_1 - 3\lambda_2 + \mu_2 = 0; \quad iii) \quad 8 - 2\lambda_1 - 2\lambda_2 + \mu_3 = 0; \\ iv) \quad & \lambda_1 g_1 = 0; \quad v) \quad \lambda_2 g_2 = 0; \quad vi) \quad \mu_1 x_1 = 0; \quad vii) \quad \mu_2 x_2 = 0; \quad viii) \quad \mu_3 x_3 = 0; \\ ix) \quad & g_i \leq 0, \quad i = 1, 2; \quad x) \quad x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

Substituting $\lambda_1 = 0$ and $\lambda_2 = 4$ into these conditions, and using the fact that $f = 4$, the system reduces to the following set of equations:

$$\begin{aligned} i) \quad & \mu_1 = 10; \quad ii) \quad \mu_2 = 6; \quad iii) \quad \mu_3 = 0; \\ iv) \quad & \lambda_1 g_1 = 0; \quad v) \quad g_2 = 0; \quad vi) \quad x_1 = 0; \quad vii) \quad x_2 = 0; \quad viii) \quad \mu_3 x_3 = 0; \\ ix) \quad & g_1 \leq 0; \quad x) \quad x_3 \geq 0; \\ xi) \quad & f = 4. \end{aligned}$$

Substituting $x_1 = x_2 = 0$ into the equality $f = 4$ yields $x_3 = \frac{1}{2}$. Hence, the maximum of the original linear program is attained at $(x_1, x_2, x_3) = (0, 0, \frac{1}{2})$.

EXERCISE 5.6 Formulate the corresponding dual linear programming problems of next LP problems. Next solve both these dual and LP problems.

- a. Maximize $U(x, y) = x + y$, subject to the constraints $x + 3y \leq 10$, $3x + y \leq 10$, $2x + 2y \leq 8$ $x \geq 0$ and $y \geq 0$.
- b. Maximize $f(x, y, z) = x - y + z$, subject to the constraints $x + y + z \leq 1$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.
- c. Minimize $f(x, y, z) = x + y + 2z$, subject to the constraints $x + y \geq 1$, $y + z \geq 2$, $x + z \geq 3$ and $x + y + 2z \geq 3$.

We conclude this section by focusing on a class of LP problems that arise in the economic literature and on the interpretation of their corresponding dual problems (see also, e.g., [27], Section 17).

EXAMPLE 5.21 (*optimal resource allocation*)

Within the standard LP problem formulation (5.2) consider a firm that produces n different products using m different inputfactors. For each inputfactor i , there is a maximum available amount b_i for production. Each inequality $a_i^T x \leq b_i$ represents the fact that producing x_j units of product j requires a_{ij} units of inputfactor i . The objective function $c^T x$ represents the firm's total profit, where c_j denotes the profit from selling one unit of product j .

From the LP theory developed above, we know that if maximum profit is achieved by producing x_i^* units of product i , $i = 1, \dots, n$, then there are m associated co-state variables λ_j^* . Moreover, each λ_j^* represents the additional profit the firm would obtain from increasing inputfactor j by one unit. Since this value does not necessarily coincide with the market price of inputfactor j , it is referred to as the shadow price of inputfactor j (see Corollary 3.19).

This allows us to interpret the dual of this problem (5.6) as follows.

The term $b^T \lambda$ represents the total (shadow) value of all production factors. Furthermore, $\sum_{i=1}^m a_{ij} \lambda_i$ corresponds to the total cost for producing one unit of product j . Since c_j is the revenue obtained from selling one unit of product j , the difference $c_j - \sum_{i=1}^m a_{ij} \lambda_i$ represents the (shadow) profit from producing one unit of product j .

Thus, the dual problem can be interpreted as minimizing the total production cost while ensuring that each product generates at least a minimal profit.

Mixed Exercises

EXERCISE 5.7 Let $b \in \mathbb{R}$. Use Lemma 5.4 to show that next sets G are convex.

- $G := \{x \in \mathbb{R}^n \mid a^T x \leq b\}$, where $a \in \mathbb{R}^n$, $a \neq 0$.
- $G := \{(x, y) \in \mathbb{R}_{++}^2 \mid \frac{x^2}{y} \leq b\}$, where $b > 0$.
- $G := \{(x, y) \in \mathbb{R}^2 \mid \log(e^x + e^y) \leq b\}$.
- $G := \{(x, y, z) \in \mathbb{R}_{+++}^3 \mid \frac{1}{x^2} \leq 1, y^3 \leq 9 \text{ and } \sqrt{z} \geq 4\}$.

EXERCISE 5.8 Solve next constrained optimization problems.

- Maximize $f(x, y, z) = x + y + z$ s.t. $x + y \leq 1$, $y + z \leq 2$, $x + z \leq 3$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.
- Minimize $f(x, y) = e^{-(x+y)}$ s.t. $x \leq 4 + y$.
- Maximize $f(x, y, z) = x + y - z$ s.t. $x + y + z \geq 1$, $x + 4y \leq 5$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.
- Minimize $f(x, y, z) = e^x + y^2$, s.t. $x^2 + y^2 + z^2 \leq 1$ and $x + y + z^2 \leq 3$.
- Maximize $f(x, y) = -(x - 1)^2 - (y - 1)^2$ on $C := \{(x, y) \mid x > 0, y > 0\}$ s.t. $x + y \leq 1$.
- Reconsider the maximization problem in item c, but now on the set $C := \{(x, y) \mid x < 0, y < 0\}$.
- Minimize $f(x, y) = (x - 1)^2 + (y - 1)^2$ s.t. $x + y \geq 3$ and $x^2 + y^2 \leq 9$.
- Maximize $f(x, y) = y - x$ s.t. $x + 2y \geq 5$, $2x + y \geq 4$ and $y + x \leq 4$.
- Minimize $f(x, y, z) = 2x + y + z$ s.t. $\ln(x + y + z) \geq 1$ and $x + y + z \geq 0$.
- Maximize $f(x_1, x_2, x_3, x_4, x_5) = x_1 + 2x_2 - 3x_3 + 4x_4 - 5x_5$ s.t. $5x_1 - 4x_2 + 3x_3 - 2x_4 - x_5 \leq 10$, $x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$ and $x_i \geq 0$.
- Maximize $f(x, y) = e^{-x-y}$ s.t. $y - x^2 \leq 0$.

EXERCISE 5.9 Show there exists a $\lambda \in (0, 1)$ such that $\lambda \ln(x) \leq (1 - \lambda)(x - 1)$ for all $x > 0$.

EXERCISE 5.10 Assume $a, b, c_1, c_2 \in \mathbb{R}$ are some given constants. Consider the maximization of $\sqrt{x_1 + x_2}$ subject to the constraints $\max\{x_1, x_2\} \leq a$ and $\frac{x_1^2}{x_2} \leq b$, $x_i \geq c_i > 0$, $i = 1, 2$. Show that if above optimization problem has a solution, its solution will provide a global maximum.

EXERCISE 5.11 Consider the problem to minimize $f(x, y) = x^2 + 4x - 2xy + y^2 - 4y$ subject to the constraints $x \geq 0$; $y \geq 0$ and $(x - 3)^2 + (y - 3)^2 \geq 4$.

- Show the constraint set is not compact.
- Show that this is not a convex optimization problem.
- Show, using Fritz-John's theorem, that the problem has an infinite number of global minima.
- Show that the problem has no local minima.

Advanced, Deepening and Broadening Exercises

EXERCISE 5.12 (Importance differentiability assumptions in Algorithm 4.9)

This exercise demonstrates that in Step 1 of Algorithm 4.9 it is important to include any points where either f or g are not differentiable too in the search for potential optima.

Consider the maximization of $f(x) = \sqrt{1-x^2}$, $x \in [-1, 1]$ subject to $g(x) = x-1 \geq 0$.

- Use the theory presented in this section to show that this problem has a maximum at $x = 1$.
- Let $X = [-1, 1]$ and $\Lambda = \{\lambda \mid \lambda \geq 0\}$. Show that the with this problem corresponding Lagrange function $L(x, \lambda) := \sqrt{1-x^2} - \lambda(1-x)$ has not a saddle point at $(1, \bar{\lambda})$, whatever $\bar{\lambda} \geq 0$ is, on the set $X \times \Lambda$.
- Compare the result obtained in item b. with Theorem 5.13. Conclude.

EXERCISE 5.13 (A function is convex iff its epigraph is a convex set).

Consider the set of points located above the graph of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. This set is called the *epigraph* of f . Formal: $\text{epi } f := \{(x, \gamma) \mid x \in \text{domain } f \text{ and } f(x) \leq \gamma\}$.

- Determine $\text{epi } f$ if $f(x) = x^2$ with $x \in (0, 2)$.
- Let $S \subset \mathbb{R}^n$ be a convex set and $f : S \rightarrow \mathbb{R}$. Show that $f(x)$ is convex on S if and only if its epigraph on S is a convex set.
- Formulate a similar result for function g if g is concave.

EXERCISE 5.14 (Separating Hyperplane Theorem)

Let $S \subset \mathbb{R}^n$ be a closed convex set. Let $y \notin S$. Below you are asked to show that there exists a plane $H : \{x \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = b\}$ that separates y and S . That is, there is a plane H such that y is located to one side of H and S to the other side. Without loss of generality we will assume that $n = 2$.

- Let $y = (0, 0)$ and $S = \{(x_1, x_2) \mid x_2 \geq x_1^2 + 1\}$. Construct geometrically a line that separates y and S .
- Consider the distance function $d(x) = \|x - y\|$, where $x \in S$. Motivate why $d(x)$ is a continuous function on S .
- Consider a closed ball B with center y and finite radius r , $B = \{(x_1, x_2) \mid (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq r\}$, where r is such that $I := B \cap S$ is nonempty. Show that I is a compact set.
- Show that $d(x)$ attains a global minimum on I . Let x_0 be the argument where $d(x)$ attains this minimum.
- Let $a := x_0 - y$ and $b := a^T x_0$. Show that $a^T y < b$ and $a^T x_0 = b$.
- Let $x \in S$. Let $x(\lambda) := \lambda x + (1 - \lambda)x_0$, $\lambda \in [0, 1]$. Note that, since S is convex, $x(\lambda) \in S$, for all $\lambda \in [0, 1]$. Motivate why $\|x(\lambda) - y\|^2 \geq \|x_0 - y\|^2$, for all $\lambda \in [0, 1]$. Use this inequality to show that $(x - x_0)^T (x_0 - y) \geq 0$.
- Use item e. to conclude that $a^T x \geq b$, for all $x \in S$.
- Assume $0 \notin S$. Show there exists a vector $a \in \mathbb{R}^n$ and $b > 0$ such that $a^T x \geq b$, for all $x \in S$.

EXERCISE 5.15 (*Proof Lemma 5.8*).

In this exercise you are asked to provide a proof for Lemma 5.8. Use the notation from this Lemma 5.8.

- Define for $x \in X$, the set $Z_x := \{z \in \mathbb{R}^m \mid f_i(x) > z_i, i = 1, \dots, m\}$. Let $Z := \cup_{x \in X} Z_x$. Show that $0 \notin Z$.
- Show that Z is a convex set.
- Show that with $z \in Z_x$ also for every $\epsilon \geq 0$, $z - \epsilon \in Z$. Also show that if ϵ_i is defined by $\epsilon_i := f_i(x) - z_i - \delta_i$, for arbitrary $\delta_i > 0$, $\tilde{z} := z + \epsilon \in Z$.
- Use Lemma 5.7 to conclude that there exists a $p \leq 0$, $p \neq 0$, such that $p^T z \geq 0$ for all $z \in Z$.
- Let $\lambda_i := \frac{p_i}{\sum_{i=1}^m p_i}$. Use item c to conclude that there exists a $\lambda \in \Lambda$, $\lambda \neq 0$, such that $\sum_{i=1}^m \lambda_i f_i(x) \leq 0$, for all $x \in X$.

EXERCISE 5.16 (*counterexample of Theorem 5.10 if Slater's condition fails*)

Consider the maximization of $f(x) = x$ subject to $g(x) = -x^2 \geq 0$.

- Use the theory presented in this section to show that this problem has a maximum at $x = 0$.
- Show that the with this problem corresponding Lagrange function $L(x, \lambda) = x - \lambda x^2$ has not a saddle point at $(0, \bar{\lambda})$, whatever $\bar{\lambda} \geq 0$ is.

EXERCISE 5.17 (*Any saddle-point of Lagrange function solves KKT conditions*)

Assume f and g_i are all differentiable on an open set $X \subset \mathbb{R}^n$. Let $\Lambda := \{\lambda \in \mathbb{R}^m \mid \lambda_i \geq 0\}$.

Show that if the Lagrange function $L(x, \lambda) = f(x) - \lambda^T(g(x) - b)$ has a saddle-point equilibrium at $(\bar{x}, \bar{\lambda}) \in X \times \Lambda$, $(\bar{x}, \bar{\lambda})$ satisfy the necessary conditions 1-3 of the Kuhn-Tucker Theorem 4.15.

EXERCISE 5.18 (*Slater's constraint qualification in Convex Programming Problems*)

Assume the concave function f and convex functions g_i are all differentiable on an open convex set $X \subset \mathbb{R}^n$.

- Let $(\bar{x}, \bar{\lambda})$ satisfy the necessary conditions 1-3 of the Kuhn-Tucker Theorem 4.15. Show that then the corresponding Lagrange function $L(x, \lambda) = f(x) - \lambda^T(g(x) - b)$ has a saddle point equilibrium at $(\bar{x}, \bar{\lambda})$.
- Assume Slater's condition (5.1) applies. Use Exercise 5.17 to show that the constrained optimization problem (4.8) has a solution if and only if conditions 1-3 of Theorem 4.15 apply. (Notice the rank condition in Theorem 4.15 is dismissed here.)

EXERCISE 5.19 (*Proof Theorem 5.9*)

Adopt the notation used in Theorem 5.9.

- Consider the inequality $L(\bar{x}, \bar{\lambda}) \leq L(\bar{x}, \lambda)$. Show that, since this equality holds $\forall \lambda \in \Lambda$, we can conclude that $g(\bar{x}) \leq b$.
- Use item a. to show that $\bar{\lambda}^T(g(\bar{x}) - b) = 0$.

- c. Use item b. and the fact that $L(x, \bar{\lambda}) \leq L(\bar{x}, \bar{\lambda})$, $\forall x \in X$, to show that $\bar{x}$ solves the constraint optimization problem.

EXERCISE 5.20 (*Proof Theorem 5.10*)

Adopt the notation used in Theorem 5.10. Assume in the items a-d below that all assumptions in the theorem apply except Slater's condition.

- Show that the intersection of the sets $\{x \in X \mid f(x) - f(\bar{x}) > 0\}$ and $\{x \in X \mid b_i - g_i(x) \geq 0\}$, $i = 1, \dots, m$, is empty.
- Use Lemma 5.8 to show that there exist coefficients $\bar{\lambda}_i \geq 0$, $i = 0, 1, \dots, m$, not all zero, such that $\bar{\lambda}_0(f(x) - f(\bar{x})) - \sum_{i=1}^m \bar{\lambda}_i(g_i(x) - b_i) \leq 0$, for all $x \in X$.
- Use item b. to show that there exist coefficients $\bar{\lambda}_i \geq 0$, $i = 0, 1, \dots, m$ (not all zero), such that with $L_0(x, \lambda) := \lambda_0 f(x) - \sum_{i=1}^m \lambda_i(g_i(x) - b_i)$, $L_0(x, \bar{\lambda}) \leq L_0(\bar{x}, \bar{\lambda})$ if $\bar{x}$ solves the constraint optimization problem.
- Use item b. to show that $\sum_{i=1}^m \bar{\lambda}_i(g_i(\bar{x}) - b_i) = 0$.
- Use item b. to show by contradiction that, if Slater's condition holds, $\bar{\lambda}_0 \neq 0$.
- Assume Slater's condition applies. Show that in that case the Lagrange function $L(x, \lambda) := f(x) - \sum_{i=1}^m \lambda_i(g_i(x) - b_i)$ has a saddle point at $(\bar{x}, \bar{\lambda})$, for some $\bar{\lambda} \in \Lambda$.

EXERCISE 5.21 (*Concave function on a convex set has at most one (global) maximum*)

Consider a concave function $f(x)$ defined on a convex set C .

- Show that if $f(x)$ attains a maximum at $\bar{x}$, $f(\bar{x})$ is a global maximum for f on S .
- Assume f is strictly concave on C . Show that if f has a global maximum at $\bar{x}$, there is no other point in C where this maximum is attained.

EXERCISE 5.22 (*First-order criterion for a maximum if concave function is differentiable*)

Assume the concave function f is differentiable on the convex set C . Below you are asked to show that f has a maximum at $\bar{x}$ on C iff $Df^T(\bar{x})(x - \bar{x}) \leq 0$ for all $x \in C$.

- Use the first order characterization of a concave function to show that if $Df^T(\bar{x})(x - \bar{x}) \leq 0$, for all $x \in C$, f has a maximum at $\bar{x}$ on C .
- Again, use the first order characterization of a concave function to show that if f has a maximum at $\bar{x}$ on C , $Df^T(\bar{x})(x - \bar{x})$ can never be positive for some $x \in C$.
- Give a geometric interpretation of the condition $Df^T(\bar{x})(x - \bar{x}) \leq 0$ for all $x \in C$.

EXERCISE 5.23 (*Proof Theorem 5.13*)

Adopt the notation used in Theorem 5.13. Let $g_i(x) := a_i^T x - b_i$, with $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, with $a_i \neq 0$, $i = 1, \dots, n$.

- Consider item b. from Exercise 5.20. Show that choosing $\bar{\lambda}_0 = 0$ here yields a contradiction.
- Follow the lines of the proof of Exercise 5.20 to conclude that the Lagrange function $L(x, \lambda) := f(x) - \sum_{i=1}^m \lambda_i(g_i(x) - b_i)$ has a saddle point at $(\bar{x}, \bar{\lambda})$, for some $\bar{\lambda} \in \Lambda$.

EXERCISE 5.24 (*Equivalent necessary and sufficient conditions for solving LP problem*)

In this exercise you are asked to prove next equivalent statement for Theorem 5.14 for the LP problem (5.2).

The LP problem (5.2) has a maximum at x if and only if next conditions apply:

- i. $c^T - \lambda^T A \leq 0$; $(c^T - \lambda^T A)x = 0$;
- ii. $\lambda_i(a_i^T x - b_i) = 0$, $i = 1, \dots, m$;
- iii. $\lambda \geq 0$; $Ax \leq b$ and $x \geq 0$.

Prove this result using Theorem 5.14.

EXERCISE 5.25

In this exercise you are asked to show that the function values of an affine function change monotonically if one evaluates them on a line.

- a. Show that one can parameterize a line l in $\mathbb{R}^n$ as: $l(t) = u + tv$, where $u, v \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
- b. Show, using item a., that the function values of the affine function $f(x) = a^T x + b$, with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, either remain constant, strictly increase or strictly decrease if one considers them on the line $l(t)$, where t increases.

EXERCISE 5.26 (*Proof items a-c of Corollary 5.15*)

Consider the LP problem (5.2). Show next properties hold.

- a. Show, using Theorem 5.14, that a solution cannot occur at the interior of the constraint set. Therefore, if the LP has a solution, it is attained at the border of the constraint set.
- b. If the LP problem is solved at $\bar{x}$ and $\bar{y}$ then it is also solved at any point on the line segment between $\bar{x}$ and $\bar{y}$. Hint: consider the parameterization of an arbitrary point on the line segment and show this point belongs to the constraint set and the objective value attained at this point is everywhere the same.

More general we have: if the LP problem has a solution at n different points, then the LP problem is also solved at any point in the convex hull of these points.

- c. The LP problem has either no, a unique or an infinite number of solutions.

EXERCISE 5.27 (*Proof item d of Corollary 5.15*)

In this exercise you are asked to provide a proof for the last statement made in Corollary 5.15. That is, in case the LP problem has a solution there will be at least one corner point where this solution is attained.

To prove this assertion assume $\bar{x}$, not a corner point, is a solution for the LP problem.

- a. Assume $n - 1$ independent constraints are binding at $\bar{x}$. Show that the line defined by these $n - 1$ binding constraints intersects another constraint $g_n(x)$ which is independent from these $n - 1$ binding constraints.
- b. Use Exercise 5.25 to show that item a. implies that there is a corner point where the LP attains its maximum.

- c. Prove statement d. in Corollary 5.15 by proceeding by induction along the lines of items a. and b. above.

EXERCISE 5.28 (*Max-Min (in)equality and saddle-point existence*)

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ and $f : X \times Y \rightarrow \mathbb{R}$. Assume all maxima and minima mentioned below exist (otherwise one may consider suprema and infima in items a. and b.).

- Show that $\underline{f} := \max_{x \in X} \min_{y \in Y} f(x, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y) =: \bar{f}$.
- Let $\bar{x} \in \arg \max_{x \in X} \min_{y \in Y} f(x, y)$ and $\bar{y} \in \arg \min_{y \in Y} \max_{x \in X} f(x, y)$. Show that for all $x \in X$ $f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y)$ for all $y \in Y$ if and only if $\underline{f} = \bar{f}$. That is, f has a saddle-point equilibrium iff. a max-min equality holds.
- Show that if $f(x, y) \in C^2$ and $f(x, y)$ is strictly convex in one variable and strictly concave in the other variable the max-min equality holds.

EXERCISE 5.29 (*Duality gap Lagrange dual function*)

Assume Problem (4.8) has a solution $f(x^*) = p$. Consider for each $\lambda \geq 0$ and $x \in C$ the with problem (4.8) associated Lagrange function $L(x, \lambda) := f(x) - \lambda^T(g(x) - b)$. Let for each fixed $\lambda \geq 0$ $L_D(\lambda) := \sup_{x \in C} L(x, \lambda)$. Let $d := \inf_{\lambda \geq 0} L_D(\lambda)$.

Show that $p \leq d$. Hint: use Exercise 5.28.

EXERCISE 5.30 (*Convexity Lagrange dual function*)

Consider the with problem (4.5) associated Lagrange function $L(x, \lambda) := f(x) - \lambda(g(x) - b)$ and, for each fixed $\lambda \geq 0$, $L_D(\lambda) := \sup_{x \in C} L(x, \lambda)$. In this exercise you are asked to prove that this Lagrange dual function, $L_D(\lambda)$, is convex.

- Show that $\sup_{x \in C} f(x) + g(x) \leq \sup_{x \in C} f(x) + \sup_{x \in C} g(x)$.
- Show that $L_D(\tau\lambda_1 + (1 - \tau)\lambda_2) \leq \tau L_D(\lambda_1) + (1 - \tau)L_D(\lambda_2)$.

EXERCISE 5.31 (*Lagrange dual function LP problem*)

Consider the with the standard LP problem associated Lagrange function $L(x, \lambda, \mu) := f(x) - \lambda^T(Ax - b) + \mu^T x$ and, for fixed $\lambda \geq 0$ and $\mu \geq 0$, the corresponding Lagrange dual function $L_D(\lambda, \mu) := \sup_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$.

- Show that $L_D(\lambda, \mu) = \begin{cases} b^T \lambda & \text{if } A^T \lambda - c - \mu = 0; \\ \infty & \text{elsewhere.} \end{cases}$
- Show that $\inf_{\lambda \geq 0, \mu \geq 0} L_D(\lambda, \mu) = \inf_{\lambda \geq 0} b^T \lambda$ subject to $A^T \lambda - c \geq 0$.

EXERCISE 5.32 (*A Quadratic programming problem with linear equality constraints*)

Consider the problem to minimize $f(x) = x^T Q x$, where Q is a positive definite matrix, subject to the linear constraints $Ax = b$. Denote its solution by p .

- Reformulate the problem into the standard form and show that the with this problem associated Lagrange function is $L(x, \lambda, \mu) = -x^T Q x - \lambda^T(Ax - b) - \mu^T(b - Ax)$ with $\lambda \geq 0$ and $\mu \geq 0$.

- b. Determine for fixed $\lambda \geq 0$ and $\mu \geq 0$ the with the standard form corresponding Lagrange dual function $L_D(\lambda, \mu) := \sup_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$. Show that the image of $L_D(\lambda, \mu)$ coincides with the image of $g(y) := \frac{1}{4}y^T AQA^T y + y^T b$.
- c. Assume matrix A has full row rank. Show that $p \geq d$, where $d = b^T(AQA^T)^{-1}b$.

EXERCISE 5.33 (*Quadratic programming problem*)

The quadratic programming problem (qpp) is concisely stated as follows: find $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^T Qx + q^T x$ subject to the linear constraints $Ax + b \geq 0$ and $x \geq 0$, where $x, b, q \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix, and the constraint set is nonempty.

- a. Show that the qpp has a solution if the constraint set is bounded.
- b. Give an example of a qpp that has no solution.
- c. Show that if Q is positive semi-definite any solution of the with this problem corresponding KKT conditions provides a global minimum.
- d. Show that if the qpp has a solution, there exist $x \geq 0$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ that satisfy the set of (in)equalities:
1. $-Qx - q + A^T \lambda_1 + \lambda_2 = 0$;
 2. $x \geq 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $Ax + b \geq 0$;
 3. $\lambda_1^T (Ax + b) = 0$ and $\lambda_2^T x = 0$.
- e. Let $C := \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$ and $d := \begin{bmatrix} q \\ b \end{bmatrix}$. Show that the advertized necessary conditions in item d. can be reformulated as that there exists a vector $z \geq 0$ such that with $w := Cz + d \geq 0$, $w^T z = 0$.
- f. Use either item d. or e. to find the solution of the problem to minimize $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 - x_2 - x_3$, subject to the constraints $x_i \geq 0$ and $x_1 + x_2 + x_3 \leq 1$.

EXERCISE 5.34 (*Equivalence linear complementarity problem and quadratic programming problem*)

Consider the so-called *linear complementarity problem* (lcp) to find a vector $x \geq 0$ satisfying $Ax + b \geq 0$ and $x^T(Ax + b) = 0$ (i.e.: either $x_i = 0$ or $(Ax + b)_i = 0$ for all i) (where $x, b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$). Also, consider the with this problem associated so-called *quadratic programming problem* (qpp): $\min_{x \in \mathbb{R}^n} x^T(\frac{1}{2}(A + A^T)x + b)$ subject to the linear constraints $Ax + b \geq 0$ and $x \geq 0$.

- a. Show that for any square matrix A , $x^T Ax = \frac{1}{2}x^T(A + A^T)x$. (Hint: consider the transpose of the scalar $x^T Ax$).
- b. Show that the solution of the linear complementarity problem (lcp) exists if and only if the quadratic programming problem (qpp) has a solution and the minimum value is 0.
- c. Consider the (lcp) problem with $b \geq 0$. Show that the (lcp) has a solution if, e.g., matrix $A + A^T$ is either positive semi-definite or has only non-negative entries.
- d. Use Exercise 5.33 to find, if possible, $x_i \geq 0$ such that $x_1(x_1 - 4x_2 + 3) = 0$ and $x_2(5x_1 + 2x_2 - 7) = 0$, where x_i satisfy $x_1 - 4x_2 \geq -3$ and $5x_1 + 2x_2 \geq 7$.

Problems with an Economic Content

EXERCISE 5.35 (optimal consumption of products)

Consider the optimal consumption x and y of two products. Assume minimal consumption of product i is c_i , $i = 1, 2$. The price of product i is p_i , $i = 1, 2$, and the maximum the consumer can spend is b which exceeds $p_1 c_1 + p_2 c_2$. Assume utility derived from consumption of one unit of product i is s_i , $i = 1, 2$.

- Formulate the optimization problem. Show that it is optimal to consume the minimal amount of product 1 and as much as possible from product 2 if $\frac{s_2}{p_2} > \frac{s_1}{p_1}$ and the other way around if this last inequality reverses.
- Assume next that the maximum available amount of product i is m_i and that the excess of the products not consumed must be recycled at a cost of d_i (with $d_i < p_i$) per unit of product, $i = 1, 2$. Assume the disutility derived of recycling a unit of product i is t_i .

Reformulate the optimization problem. Show that the problem has only a solution iff. the budget b exceeds the sum of the cost of buying the minimal amount of the products (s)he needs and the cost of recycling the rest of the products. Solve the problem under this premise. Show that again the optimal solution is to spend as much as possible on one product and if the maximum quatum of this product is reached, spend the rest of the budget on the other product. Introduce the "shadow" prices $\tilde{p}_i := p_i - d_i$ and "shadow" preferences $\tilde{s}_i := s_i + t_i$, $i = 1, 2$. Show that again the sign of $\frac{\tilde{s}_1}{\tilde{p}_1} - \frac{\tilde{s}_2}{\tilde{p}_2}$ determines the location of the maximum.

EXERCISE 5.36 (optimal portfolio selection)

Consider an investor who has to make a decision on which fraction ω_b from his capital X he should invest in bonds, the fraction ω_s he should invest in stocks and the remaining fraction $1 - \omega_b - \omega_s$ he should put on his savings account. For this portfolio the expected returns μ are $(1 - \omega_b - \omega_s)(1 + r)X + \omega_b(1 + r_b)X + \omega_s(1 + r_s)X$ and the associated variance $\sigma^2 = (1 - \omega_b - \omega_s)^2 \sigma_0^2 + \omega_b^2 \sigma_b^2 + \omega_s^2 \sigma_s^2$. Here r (σ_0^2), r_b (σ_b^2) and r_s (σ_s^2) are the expected returns (variance) for his saving accounts, bonds and stocks, respectively. Assume $r \leq r_b \leq r_s$ and $0 \leq \sigma_0^2 < \sigma_b^2 < \sigma_s^2$.

To make a choice in this decision problem the investor uses a utility function $U(\mu, \sigma) = \mu - \alpha \sigma^2$, where $\alpha \geq 0$ expresses his attitude towards risk, measured by σ^2 . His optimal portfolio is determined as the fractions that maximize this utility. However, he does not want to invest more than a fraction $\bar{b}$ into bonds, a fraction $\bar{s}$ into stocks (where $\bar{b} \geq \bar{s}$) and the sum of these fractions should be at most $\bar{t}$, where $\bar{t} < \bar{b} + \bar{s}$.

- Formulate the optimization problem.
- Show this is a convex optimization problem with linear constraints.
- Show that the solution is attained at the interior of the constraint set iff. with $d := (\sigma_0^2 + \sigma_s^2)(\sigma_0^2 + \sigma_b^2) - \sigma_0^4$ next conditions apply:
 - $0 < 2\alpha\sigma_0^2\sigma_s^2 + ((r_b - r)\sigma_s^2 - (r_s - r_b)\sigma_0^2)X < 2\alpha d\bar{b}$;
 - $2\alpha\sigma_0^2\sigma_b^2 + ((r_s - r)\sigma_b^2 + (r_s - r_b)\sigma_0^2)X < 2\alpha d\bar{s}$; and,

$$\text{iii)} \quad 2\alpha\sigma_0^2(\sigma_b^2 + \sigma_s^2) + ((r_b - r)\sigma_s^2 + (r_s - r)\sigma_b^2)X < 2\alpha d\bar{t}.$$

Maximum utility is then attained at:

$$(\omega_b, \omega_s) = \frac{1}{2\alpha d}(2\alpha\sigma_0^2\sigma_s^2 + ((r_b - r)\sigma_s^2 - (r_s - r_b)\sigma_0^2)X, 2\alpha\sigma_0^2\sigma_b^2 + ((r_s - r)\sigma_b^2 + (r_s - r_b)\sigma_0^2)X).$$

- d. Assume $0 < r < r_b$. Show that if the maximum occurs on the constraint set then it is at either one of next locations:

$$(\omega_b, \omega_s) = (\bar{b}, \frac{1}{2\alpha(\sigma_0^2 + \sigma_s^2)}((r_s - r)X + 2\alpha\sigma_0^2(1 - \bar{b}))) \text{ provided } \omega_s \leq \bar{t} - \bar{b} \text{ and } (r_b - r)X - 2\alpha\bar{b}\sigma_b^2 + 2\alpha\sigma_0^2(1 - \bar{b} - \omega_s) \geq 0.$$

$$(\omega_b, \omega_s) = (\frac{1}{2\alpha(\sigma_0^2 + \sigma_b^2)}((r_b - r)X + 2\alpha\sigma_0^2(1 - \bar{s})), \bar{s}) \text{ provided } \omega_b \leq \bar{t} - \bar{s} \text{ and } (r_s - r)X - 2\alpha\bar{s}\sigma_s^2 + 2\alpha\sigma_0^2(1 - \bar{s} - \omega_b) \geq 0.$$

$$(\omega_b, \omega_s) = \frac{1}{2\alpha(\sigma_b^2 + \sigma_s^2)}((r_b - r_s)X + 2\alpha\sigma_s^2\bar{t}, (r_s - r_b)X + 2\alpha\sigma_b^2\bar{t}) \text{ provided } \omega_b \leq \bar{b}, \omega_s \leq \bar{s} \text{ and } (r_b - r)X + 2\alpha\sigma_0^2(1 - \bar{t}) - 2\alpha\sigma_b^2\omega_b \geq 0.$$

- e. Assume $\sigma_0 = 0$. Show that at an interior solution the fractions of cash, bonds and stocks are:

$$1 - \frac{r_b - r}{2\alpha\sigma_b^2}X - \frac{r_s - r}{2\alpha\sigma_s^2}X, \frac{r_b - r}{2\alpha\sigma_b^2}X \text{ and } \frac{r_s - r}{2\alpha\sigma_s^2}X.$$

Show that for large choices of α an interior solution occurs if $r_b > r$. Provide an interpretation of α .

- f. Assume $\sigma_0 = 0$; $\bar{t} > 1$ and α satisfies next inequalities: $\alpha > \frac{1}{\bar{b}} \frac{r_b - r}{2\sigma_b^2}X$; $\alpha > \frac{1}{\bar{s}} \frac{r_s - r}{2\sigma_s^2}X$; and $\frac{1}{\bar{t}}(\frac{1}{2} \frac{r_b - r}{\sigma_b^2}X + \frac{1}{2} \frac{r_s - r}{\sigma_s^2}X) < \alpha < \frac{1}{2} \frac{r_b - r}{\sigma_b^2}X + \frac{1}{2} \frac{r_s - r}{\sigma_s^2}X$. Determine the optimal solution for the problem and interpret it.

- g. Assume $\sigma = 0$. Show that no investment in bonds occurs and maximal investment in stocks iff. $r_b = r$ and the investor is risk seeking. Quantify the notion of risk seeking here.

EXERCISE 5.37 (production under delivery constraints)

Consider a price taking producer who manufactures 2 products. He has a contract to supply for product i a minimum of $\bar{y}_i$, $i = 1, 2$. The maximum amount of labor and capital he can get for the total production of both goods is $\bar{L}$ and $\bar{K}$, respectively. Assume the production functions of the goods are specified by $y_i = K_i^{\alpha_i} L_i^{1-\alpha_i}$, $i = 1, 2$, with $\alpha_i \in (0, 1)$. Assume the cost of labor and capital is per unit ω and r , respectively. Assume the producer wants to maximize his profit and that the price he gets for product i is p_i , $i = 1, 2$.

- Formulate the optimization problem.
- Show this is a convex optimization problem with linear constraints.
- Show that the necessary and sufficient conditions for a maximum are obtained as the solutions of next set of equations:
 - $\alpha_1 y_1(p_1 + \lambda_1) = K_1(r + \lambda_4)$; ii) $\alpha_2 y_2(p_2 + \lambda_2) = K_2(r + \lambda_4)$; iii) $(1 - \alpha_1)y_1(p_1 + \lambda_1) = L_1(\omega + \lambda_3)$; iv) $(1 - \alpha_2)y_2(p_2 + \lambda_2) = L_2(\omega + \lambda_3)$;
 - With $g_i := \bar{y}_i - y_i$, $i = 1, 2$; $g_3 := L_1 + L_2 - \bar{L}$ and $g_4 := K_1 + K_2 - \bar{K}$, $\lambda_i g_i = 0$, $i = 1, \dots, 4$;
 - $\lambda_i \geq 0$ and $g_i \leq 0$, $i = 1, \dots, 4$.
- Show that the problem has almost never a solution at the interior of the constraint set or at a point where precisely one constraint is binding.

- e. Consider the case $\alpha_1 = \alpha_2$ and $p_1 > p_2$. Show, by contradiction, that the constraint g_2 is then always binding. That is the producer will produce precisely the amount of product two he is obliged too.
- f. Show that the case that both production constraints are simultaneously binding occurs iff. prices satisfy $p_1 \leq \left(\frac{r}{\alpha_1}\right)^{\alpha_1} \left(\frac{\omega}{1-\alpha_1}\right)^{1-\alpha_1}$ and $p_2 \leq \left(\frac{r}{\alpha_2}\right)^{\alpha_2} \left(\frac{\omega}{1-\alpha_2}\right)^{1-\alpha_2}$. Explain this result.
- g. Assume $\alpha_1 \neq \alpha_2$. Show that the case that both production factor constraints are simultaneously binding occurs iff. the prices of the production factors satisfy $\omega \leq \left(\frac{p_2^{\alpha_1}}{p_1^{\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}} \omega_L$ and $r \leq \left(\frac{p_1^{1-\alpha_2}}{p_2^{1-\alpha_1}}\right)^{\frac{1}{\alpha_1-\alpha_2}} r_K$. Here $\omega_L = \left(\frac{\alpha_2^{\alpha_1\alpha_2}(1-\alpha_2)^{\alpha_1(1-\alpha_2)}}{\alpha_1^{\alpha_1\alpha_2}(1-\alpha_1)^{(1-\alpha_1)\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}$ and $r_K = \left(\frac{\alpha_1^{\alpha_1(1-\alpha_2)}(1-\alpha_1)^{(1-\alpha_1)(1-\alpha_2)}}{\alpha_2^{(1-\alpha_1)\alpha_2}(1-\alpha_2)^{(1-\alpha_1)(1-\alpha_2)}}\right)^{\frac{1}{\alpha_1-\alpha_2}}$.

Does this solution always make sense?

It can be shown that both ω_L and r_K are located between zero and one, where r_K is close to zero and ω_L close to one if both α_1 and α_2 are close to zero. Explain this last result intuitively if $p_1 = p_2$.

CHAPTER 6

Optimization problems with more than one decisionmaker

As noted in the introduction of this monograph, a key issue in economics is developing appropriate models to describe how economic agents behave and interact. In the previous chapters, we reviewed essential tools for finding the optimum of a (constrained) optimization problem. If an economic agent has complete information about the objective function he seeks to optimize, he can use these tools to determine the values of the variables that maximize his objective.

However, in many cases, the decisions made by one agent influence the variables that another agent wishes to control optimally. Consequently, agents cannot independently set all the variables of interest. Many real-world situations involve multiple agents, each with instruments at their disposal that allow them to influence key economic variables. Each agent typically seeks to optimize different target values, which often conflict. The question then arises: which values of the variables will actually materialize?

Answering this question is rarely straightforward. It requires carefully estimating how the involved agents will respond. Agents may collaborate, partially or fully, to optimize their objectives. Conversely, such cooperation may be physically impossible or undesirable, leading to a *noncooperative* setting. In these cases, assumptions about the expected behavior of other agents become crucial in determining each agent's choice of instruments. Furthermore, in this context, uncertainty about variables or the environment can also be interpreted as a malevolent player.

If the same decision problem is revisited at a later time, after the previous situation has played out, an agent is likely to incorporate the past reactions of others when adjusting his instruments. If this process is repeated continuously, an important question arises: will the instrument values eventually converge, or will they keep oscillating?

When the values converge, they share a defining property: in this equilibrium state, no agent has an incentive to unilaterally change his instrument values, given the choices of the other agents. Such a situation, in which no agent wishes to deviate unilaterally, is known as a *Nash* equilibrium.

There is an extensive body of literature that studies situations in which multiple agents influence the outcome of a process that each seeks to control. This field is known as *game theory*. In this chapter, we will focus on three types of play that are commonly examined in this literature: situations in which all agents cooperate; situations in which none of the agents cooperates with others; and situations in which one agent holds a leading or dominant position in the decision-making process. We do not aim to provide a comprehensive treatment of the concepts used in this field. The presentation below is intended only as a brief introduction. Readers seeking a deeper understanding are referred to the literature, for example [1], [2], [16], [20], and [30].

6.1 Full Cooperation: the Pareto Frontier

In economics, many situations are characterized either by a single agent who seeks to optimize multiple criteria, or by multiple agents, each with their own objectives, whose outcomes depend on the actions of the others. To achieve the best possible results, these agents may choose to cooperate and make collective decisions regarding their set of actions.

We illustrate both of these situations with an example.

EXAMPLE 6.1 Consider a person who works to earn an income but also wants to enjoy some leisure time, y , for recreation, while getting at least 8 hours of sleep each day. Assume that his income, I , depends linearly on the number of hours worked, x , at a rate of €15 per hour. That is,

$$I = 15x.$$

Additionally, suppose he must earn at least €80 per day to cover fixed costs such as housing, insurance, and daily expenses.

Define the objective functions

$$J_1(x, y) := 15x \text{ and } J_2(x, y) := y - 8.$$

On a daily basis, this person faces the problem of maximizing both J_1 and J_2 with respect to x and y , subject to the constraints

$$x + y = 24, \quad 15x \geq 80 \text{ and } y \geq 8.$$

EXAMPLE 6.2 Consider two people who need to divide €100 between them. Let x denote the amount received by person 1, and y the amount received by person 2. Define the objective functions

$$J_1(x, y) := x \text{ and } J_2(x, y) := y.$$

These individuals then face the problem of maximizing both J_1 and J_2 with respect to x and y , subject to the constraints

$$x + y \leq 100, \quad x \geq 0 \text{ and } y \geq 0.$$

Clearly, in both examples, there is no single solution that can be considered "the best." For instance, in Example 6.1, the person could work 16 hours and sleep the remaining 8 hours, earning an income of €240 but having no leisure time. Alternatively, he could work just 5 hours and 20 minutes - enough to cover his daily fixed costs - and spend the rest of the day at leisure. In fact, any number of working hours between these extremes can be considered a valid solution, as long as no additional criterion is specified to compare the two objectives.

A similar conclusion holds for Example 6.2. The solution in which person 1 receives €100 and person 2 receives nothing is just as valid as the solution where person 2 receives all the money and person 1 receives nothing. However, it is also clear that if both persons were to agree on receiving only €40 each, this would not be optimal, since the position of one or both individuals could be improved without disadvantaging the other.

In the remainder of this chapter, we will focus on identifying those allocations of instruments for which the only way to make one individual better off is to make another worse off. Such allocations are referred to as *Pareto efficient*. Also, when a person must decide how to optimally allocate his (her) instruments to achieve multiple objectives, this approach appears reasonable: allocating the instruments in such a way that none of the objectives can be improved without worsening at least one of the others.

Formally, let $x_i \in \mathbb{R}^{m_i}$ be the (set of) instrument(s) available to agent i , and let Γ represent the set of all possible allocations of the instruments $x := (x_1, \dots, x_n)$. Suppose that Γ is characterized by the following system of (in)equalities

$$\Gamma := \{x \in C \mid g_i(x) \leq b_i, \quad i = 1, \dots, k, \text{ and } h_j(x) = c_j, \quad j = 1, \dots, n\}, \text{ with } m = \sum_{i=1}^N m_i, \quad (6.1)$$

where $J_i(x)$ denotes the objective that agent i wishes to maximize¹⁹. All functions J_i , g_i and h_i are all defined on an open subset $C \subset \mathbb{R}^n$, for $i = 1, \dots, N$.

¹⁹ For simplicity, we assume here that each player has only a single criterion to optimize. This framework can, of course, be easily extended to accommodate multiple criteria for each player.

DEFINITION 6.3 A point $\hat{x} \in \Gamma$ is called *Pareto efficient*²⁰ if there exists no $x \in \Gamma$ such that

$$J_i(x) \geq J_i(\hat{x}), \quad i = 1, \dots, N,$$

with at least one of the inequalities being strict.

The corresponding point $(J_1(\hat{x}), \dots, J_N(\hat{x})) \in \mathbb{R}^N$ is called a *Pareto solution*, and the collection of all Pareto solutions is referred to as the *Pareto frontier*.

The concept of Pareto efficiency is straightforward: a solution is Pareto efficient if it is impossible to improve the situation of one person without making someone else worse off. A *Pareto improvement* occurs when a different allocation of the instruments leads to a net gain for at least one agent, while no one is disadvantaged by the change.

A Pareto solution is therefore never "dominated" and is also referred to as an *undominated solution*. In most cases, there is more than one Pareto solution. Figure 6.1 illustrates the Pareto frontier for Example 6.1.

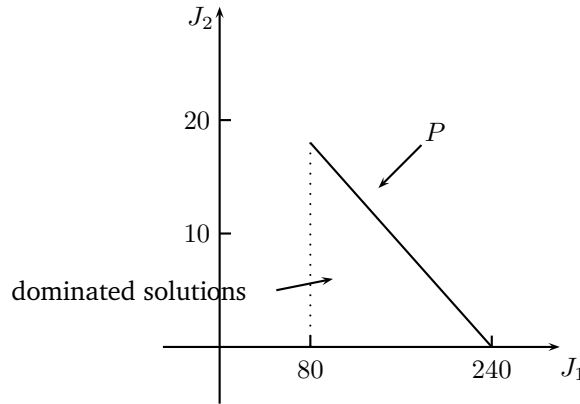


FIGURE 6.1 – Pareto frontier P , Example 6.1.

A Pareto optimal outcome does not necessarily imply that it is fair or equitable. For example, consider a situation in which a single person owns all of the world's resources. To improve the welfare of anyone else, some resources would have to be taken from this person, making him (her) worse off. Thus, identifying the Pareto frontier is one thing, but selecting a specific point on it can be more challenging. For instance, technological improvements may increase overall economic welfare but cause some people to lose their jobs. This does not constitute a Pareto improvement. However, if those who lose their jobs are compensated through welfare benefits, it becomes possible for society as a whole to benefit from the technological advances while ensuring that no one is worse off. Determining a "fair" solution on the Pareto frontier is a topic addressed by bargaining theory in game

²⁰ In the literature it is also said that $\hat{x}$ provides a *vector (global) maximum* of $J(x) = [J_1(x), \dots, J_N(x)]$.

theory. Clearly, the notion of fairness is subjective. For further details on this literature, see, for example, [29].

In the two subsections below, we discuss sufficient and necessary conditions for identifying Pareto efficient solutions. We begin with the sufficient conditions, as they are easier to derive and can be readily connected to the necessary conditions, which we will address subsequently.

6.1.1 Sufficient Conditions for Pareto Efficiency

It turns out that there is a straightforward method to identify - often - a large subclass of Pareto solutions. In this approach, the following set of parameters, known as the *unit simplex* $\mathcal{A}$, plays a central role.

$$\mathcal{A} := \{\alpha = (\alpha_1, \dots, \alpha_N) \mid \alpha_i \geq 0 \text{ and } \sum_{i=1}^N \alpha_i = 1\}.$$

Let $\alpha_i \in (0, 1)$ with $\sum_{i=1}^N \alpha_i = 1$, and $\hat{x} \in \Gamma$ be a point that maximizes

$$\sum_{i=1}^N \alpha_i J_i(x).$$

If $\hat{x}$ were not Pareto efficient, then by definition there would exist some $\bar{x} \in \Gamma$ such that

$$J_i(\bar{x}) \geq J_i(\hat{x}), \quad i = 1, \dots, N,$$

with at least one of the inequalities being strict. But this would imply

$$\sum_{i=1}^N \alpha_i J_i(\bar{x}) > \sum_{i=1}^N \alpha_i J_i(\hat{x}),$$

which contradicts the assumption that $\hat{x}$ maximizes the weighted sum.

Therefore, any point that maximizes a strict weighted average of the agent's objectives yields a Pareto efficient allocation. Theorem 6.4 below summarizes this result.

THEOREM 6.4 *Let $\alpha_i \in (0, 1)$, with $\sum_{i=1}^N \alpha_i = 1$. If $\hat{x} \in \Gamma$ is such that*

$$\hat{x} \in \arg \max_{x \in \Gamma} \left\{ \sum_{i=1}^N \alpha_i J_i(x) \right\}, \quad (6.2)$$

then $\hat{x}$ is Pareto efficient.

Notice that if the objective functions $J_i(x)$, $i = 1, \dots, N$, are continuous, then the function

$$J(x) = \sum_{i=1}^N \alpha_i J_i(x)$$

is also continuous for any scalars $\alpha_i \in \mathbb{R}$. Consequently, if the constraint set Γ is compact, Weierstrass' theorem ensures that $J(x)$ attains a maximum for every $\alpha \in \mathcal{A}$.

From this, the following corollary follows immediately from Theorem 6.4.

COROLLARY 6.5 *Assume the constraint set Γ is compact and $J_i(x)$ $i = 1, \dots, N$, are continuous. Then, for all $\alpha \in \mathcal{A}$ with $\alpha_i > 0$, $\hat{x}$ in (6.2) exists and yields a Pareto efficient solution.*

Now consider the Lagrangian associated with the above maximization problem (for a constraint set Γ that is not necessarily compact):

$$L(x, \lambda, \mu) := \sum_{i=1}^N \alpha_i J_i - \sum_{i=1}^k \lambda_i (g_i(x) - b_i) - \sum_{i=1}^m \mu_i (h_i(x) - c_i).$$

Using Theorem 6.4 (see Exercise 6.25), one can readily verify the following result, which is closely related to the necessary conditions stated in Theorem 6.11 below.

THEOREM 6.6 *Assume there exist $\hat{\alpha}_i \in (0, 1)$, with $\sum_{i=1}^N \hat{\alpha}_i = 1$, $\hat{\lambda}_i \geq 0$, $\hat{\mu}_i$ and $\hat{x} \in \Gamma$ such that $\hat{\lambda}_i (g_i(\hat{x}) - b_i) = 0$ and $L(x, \hat{\lambda}, \hat{\mu}) \leq L(\hat{x}, \hat{\lambda}, \hat{\mu})$, $\forall x \in \Gamma$. Then $\hat{x}$ is Pareto efficient.*

By imposing additional concavity assumptions on the objective functions, it is possible to identify almost all Pareto solutions using this approach. Theorem 6.7 below states that, under these assumptions, every Pareto efficient strategy can be obtained by solving the maximization problem (6.2) over the set $\mathcal{A}$.

Since, by Theorem 6.4, the instruments that maximize $\sum_{i=1}^N \alpha_i J_i(x)$ for all strictly positive α_i are already Pareto efficient, it remains only to verify the Pareto efficiency of solutions that maximize $\sum_{i=1}^N \alpha_i J_i(x)$ on the boundary of $\mathcal{A}$ to capture all Pareto solutions in this case. The proof of this theorem follows directly from Lemma 5.8 (see Exercise 6.26).

THEOREM 6.7 *Assume the constraint set of instruments, Γ , is convex. Moreover, assume the objective function J_i player i wants to maximize is concave, $i = 1, \dots, N$. Then, if $\hat{x}$ is Pareto efficient, there exist $\alpha \in \mathcal{A}$, such that for all $x \in \Gamma$*

$$\sum_{i=1}^N \alpha_i J_i(x) \leq \sum_{i=1}^N \alpha_i J_i(\hat{x}).$$

EXAMPLE 6.8 Consider a problem in which two agents choose to cooperate in order to achieve their objectives. Assume agent 1 seeks to maximize $J_1 = -(x-1)^2 - y^2 + 2$ w.r.t. x , while agent 2 aims to maximize $J_2 = -x^2 - (y-1)^2 + 2$ w.r.t. y .

Since there are no restrictions on the instruments, the constraint set is $\Gamma = \mathbb{R}^2$, which is convex. Furthermore, it can be easily verified that $J_i(x, y)$, $i = 1, 2$, are concave in (x, y) . Therefore, by Theorem 6.7, all Pareto solutions can be determined by maximizing the parameterized problem for each $\alpha \in \mathcal{A}$

$$J(\alpha) := -\alpha(x-1)^2 - \alpha y^2 - (1-\alpha)x^2 - (1-\alpha)(y-1)^2 + 2.$$

Straightforward differentiation shows that

$$DJ(\alpha) = [2\alpha - 2x, 2(1-\alpha) - 2y].$$

So, for every $\alpha \in \mathcal{A}$ there is a unique stationary point $(\hat{x}, \hat{y}) = (\alpha, 1-\alpha)$. Note that $HJ(\alpha)$ is negative definite. Therefore, $J(\alpha)$ has a maximum at $(\hat{x}, \hat{y})$ and, consequently, $(\hat{x}, \hat{y})$ is a Pareto efficient solution for $\alpha \in (0, 1)$. One easily verifies that also at $\alpha = 0$ and $\alpha = 1$ the resulting solutions are Pareto optimal.

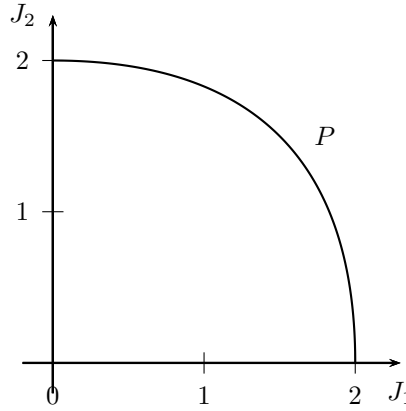


FIGURE 6.2 – The Pareto frontier P for Example 6.8.

Substitution into J_1 and J_2 yields then the Pareto solution $(J_1, J_2) = (2 - 2(1-\alpha)^2, 2 - 2\alpha^2)$. In Figure 6.2 we plotted for all $\alpha \in \mathcal{A}$ the set of Pareto solutions, yielding the Pareto frontier P .

Theorem 6.7 indicates that, under the stated assumptions, the Pareto frontier is typically an $N - 1$ dimensional surface in $\mathbb{R}^N$. This, arises because $\mathcal{A}$ is defined by the equation $\alpha_N = 1 - \alpha_1 - \dots - \alpha_{N-1}$, where the variables $\alpha_1, \dots, \alpha_{N-1}$, can to a certain extent, be chosen independently. However, the following example illustrates that this intuition does not always hold.

EXAMPLE 6.9 Consider $J_i(x_1, x_2) = -x_1^2 - x_2^2$ for $i = 1, 2$, with $\Gamma = \mathbb{R}^2$. Clearly, J_i is concave for both $i = 1, 2$. Thus, by Theorem 6.7, the set of all Pareto efficient instruments can be obtained by considering all solutions $(\hat{x}_1, \hat{x}_2)$ that maximize the following function for a fixed $\alpha \in [0, 1]$:

$$\alpha J_1 + (1 - \alpha) J_2.$$

Since $\alpha J_1 + (1 - \alpha) J_2 = -x_1^2 - x_2^2$, for all $\alpha \in [0, 1]$, it follows that there is just one Pareto efficient solution, i.e. $(\hat{x}_1, \hat{x}_2) = (0, 0)$. So, in this example there is just one Pareto solution, i.e. $(J_1(0, 0), J_2(0, 0)) = (0, 0)$, which is not a 1-dimensional surface in $\mathbb{R}^2$.

EXERCISE 6.1 Reconsider Example 6.2. Use Theorem 6.7 to show that the Pareto frontier is given by $\{(x, y) \mid x + y = 100, x \geq 0, y \geq 0\}$.

EXERCISE 6.2 Consider $J_1(x_1, x_2) = -x_1^2 - x_2^2$ and $J_2(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2$ where $\Gamma = \mathbb{R}^2$. Calculate the Pareto frontier for this example. Plot the graph of it.

EXERCISE 6.3 Let $p = 100 - x_1 - x_2$, $\pi_1(x_1, x_2) = px_1 - (30x_1 + 150)$ and $\pi_2(x_1, x_2) = px_2 - (35x_2 + 50)$. Assume agent i , $i = 1, 2$, wants to maximize π_i w.r.t. x_i , subject to the constraints $x_i \geq 0$, $p \geq 0$ and $\pi_i \geq 0$.

- Use your graphic calculator to make a plot of the set of potential allocations for the instruments (x_1, x_2) .
- Find for all $\alpha \in (0, 1)$ the maximum of $\alpha \pi_1(x_1, x_2) + (1 - \alpha) \pi_2(x_1, x_2)$.
- Plot for the set of potential allocations the solution set $(\pi_1(x_1, x_2), \pi_2(x_1, x_2))$. Indicate the Pareto frontier in this plot.
- Does the set of Pareto solutions you found in item b coincide with the Pareto frontier you found in item c? Expound on the result.

6.1.2 Necessary Conditions for Pareto Efficiency

In this section we present necessary conditions for a set of instruments $(x_1, \dots, x_N)$ to be Pareto efficient. To motivate these conditions, consider the two-player case where player i can choose values for $x_i \in \mathbb{R}$ and seeks to maximize $J_i(x_1, x_2)$, $i = 1, 2$.

Suppose (x_1^*, x_2^*) is Pareto efficient, with the corresponding Pareto solution (J_1^*, J_2^*) . Next, consider the restricted set of instruments

$$\Gamma_1 := \{(x_1, x_2) \mid J_2(x_1, x_2) \geq J_2^*\}.$$

The grey area in Figure 6.3 illustrates the set of outcomes $(J_1(x_1, x_2), J_2(x_1, x_2))$ corresponding to $(x_1, x_2) \in \Gamma_1$. From this graph, it is clear that when $J_1(x_1, x_2)$ is restricted to Γ_1 , J_1 attains a global maximum J_1^* at (x_1^*, x_2^*) .

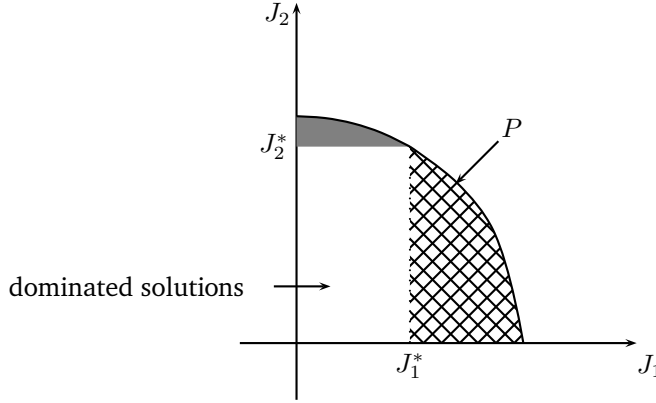


FIGURE 6.3 – Grey area: values of (J_1, J_2) with $(x_1, x_2) \in \Gamma_1$.

In a similar manner, we can consider the restricted set of instruments

$$\Gamma_2 := \{(x_1, x_2) \mid J_1(x_1, x_2) \geq J_1^*\}.$$

The crosshatch area in Figure 6.3 illustrates the set of outcomes $(J_1(x_1, x_2), J_2(x_1, x_2))$ corresponding to $(x_1, x_2) \in \Gamma_2$. From this part of the graph, it is evident that, when $J_2(x_1, x_2)$ is restricted to Γ_2 , J_2 it attains a global maximum J_2^* at (x_1^*, x_2^*) .

This motivates the following Lemma 6.10.

LEMMA 6.10 $x^* \in \Gamma$ is Pareto efficient if and only if, for all i , x^* maximizes $J_i(x)$ on the constrained set

$$\Gamma_i := \{x \in \Gamma \mid J_j(x) \geq J_j(x^*), j = 1, \dots, N, j \neq i\}, \quad \text{for } i = 1, \dots, N.$$

Lemma 6.10 demonstrates that Pareto solutions can be obtained by solving a constrained optimization problem. For example, using Theorem 4.35, this leads, in the case of player one, to the consideration of the Lagrangian:

$$L := \alpha_1 J_1(x) - \sum_{i=2}^N \alpha_i (J_i^* - J_i) - \sum_{i=1}^k \lambda_i (g_i(x) - b_i) - \sum_{i=1}^m \mu_i (h_i(x) - c_i).$$

The corresponding first-order conditions derived from Theorem 4.35 then coincide with the first-order criterion stated in Theorem 6.11 below.

THEOREM 6.11 Assume J_i , $i = 1, \dots, N$, g_i , $i = 1, \dots, k$, and h_i , $i = 1, \dots, m$, are continuous differentiable. Let $x^* \in \Gamma$ be Pareto efficient for J_i , $i = 1, \dots, N$. Consider

$$L := \sum_{i=1}^N \alpha_i J_i(x) - \sum_{i=1}^k \lambda_i (g_i(x) - b_i) - \sum_{i=1}^m \mu_i (h_i(x) - c_i).$$

Then, there exist scalars $\alpha_i \geq 0$, $i = 1, \dots, N$, $\lambda_i \geq 0$, $i = 1, \dots, k$, and μ_i , $i = 1, \dots, m$, such that $(x^*, \lambda_1, \dots, \mu_m)$ satisfy the next set of (in)equalities

1. $DL_x(x^*, \lambda_1, \dots, \mu_m) = [0 \dots 0]$.
2. $\lambda_i(g_i(x^*) - b_i) = 0$, $i = 1, \dots, k$.
3. $g_i(x^*) \leq b_i$, $i = 1, \dots, k$ and $h_i(x^*) = c_i$, $i = 1, \dots, m$.
4. $(\alpha_1, \dots, \alpha_N, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m) \neq (0, \dots, 0)$.

Applying Kuhn-Tucker's Theorem, it follows directly from Lemma 6.10 that, under certain additional regularity conditions, the parameters α_i in Theorem 6.11 are always strictly positive. This leads to the following result.

THEOREM 6.12 Consider the same notation and conditions as in Theorem 6.11. Moreover, define

$$J_{-i} := [J_1, \dots, J_{i-1}, J_{i+1}, \dots, J_N].$$

Let $x^* \in \Gamma$ be Pareto efficient for J_i , $i = 1, \dots, N$. Assume g_i , $i = 1, \dots, s$, are binding at x^* (and $g_j(x^*) < 0$ for all other inequalities), and let $g := [g_1, \dots, g_s]$.

If $\text{rank } [DJ_{-i}; Dg] = n$ at x^* for each $i = 1, \dots, N$, then there exist scalars $\alpha_i > 0$, $i = 1, \dots, N$, $\lambda_i \geq 0$, $i = 1, \dots, k$, and μ_i , $i = 1, \dots, m$, such that $(x^*, \lambda_1, \dots, \mu_m)$ satisfies the set of (in)equalities 1-4 of Theorem 6.11.

As a special case of Theorem 6.11, in the unconstrained optimization setting, the search for Pareto solutions can once again be reduced to the unit simplex. We state this result as Corollary 6.13 below (see Exercise 6.27 for a proof).

COROLLARY 6.13 Assume J_i , $i = 1, \dots, N$, are continuous differentiable on an open set $C \subset \mathbb{R}^n$. Let $x^* \in C$ be Pareto efficient for J_i , $i = 1, \dots, N$. Consider

$$L := \sum_{i=1}^N \alpha_i J_i(x).$$

Then, there exist scalars $\alpha \in \mathcal{A}$ such that $DL_x(x^*, \alpha_1, \dots, \alpha_N) = [0 \dots 0]$.

REMARK 6.14

1. Note that the necessary conditions stated in Theorem 6.11 coincide with the first-order conditions for maximizing a weighted average of the players' objectives over the set Γ ; that is, the first-order conditions of the problem

$$\max_{x \in \Gamma} \sum_{i=1}^N \alpha_i J_i(x), \text{ where } \alpha_i \geq 0.$$

Furthermore, as noted in Theorem 6.4, any solution to this maximization problem with strictly positive weights $\alpha_i > 0$, $i = 1, \dots, N$, is Pareto efficient. Moreover, Corollary 6.5 shows that if the constraint set Γ is compact, a Pareto efficient solution exists for every combination of strictly positive α_i . These observations can be useful when attempting to determine the Pareto frontier in specific situations.

2. Consider the case where the constraint set Γ contains no equality constraints. From the necessary conditions in Theorem 6.11, it follows that if an interior solution $\bar{x}$ of the constrained set is Pareto efficient, then the derivative vectors $DJ_i(\bar{x})$ must be linearly dependent.

In other words, the matrix

$$G(\bar{x}) := [D^T J_1(\bar{x}) \cdots D^T J_N(\bar{x})]$$

cannot have full column rank. This is because, for an interior solution $\bar{x}$, the necessary conditions require that all λ_i parameters vanish. As a result, item 1 in Theorem 6.11 reduces to

$$G\alpha = 0.$$

If G had full column rank, this would imply $\alpha = 0$, which violates item 4 of Theorem 6.11. Therefore, $G(\bar{x})$ must be singular.

This property is often useful in identifying Pareto efficient solutions (see, e.g., Algorithm 6.17 below). To verify the singularity of G , one can, for instance, check whether the determinant of G (if square) or of $G^T G$ equals zero.

Based on item 1 of Remark 6.14, one might be tempted to believe that all Pareto solutions can be obtained by considering the maxima of some weighted average of the players' objectives. However, this is not always the case. The following example illustrates this: for every choice of weights, no maximum exists for the weighted sum, yet there are an infinite number of Pareto solutions.

EXAMPLE 6.15 Consider the maximization of the objectives

$$J_1(x_1, x_2) = x_1 + x_2 \text{ and } J_2(x_1, x_2) = -(x_1 + x_2)^3,$$

where x_i denotes the instrument of player i , $i = 1, 2$.

The set of attainable objective values in the (J_1, J_2) plane is described by the curve

$$J_2 = -J_1^3.$$

Every point (J_1, J_2) on this curve is Pareto optimal. Consequently, the Pareto frontier consists of all points in the (J_1, J_2) -plane lying on the curve $J_2 = -J_1^3$.

Note that every point $(x_1, x_2) \in \mathbb{R}^2$ is Pareto efficient, and that all pairs of the form $(x_1, c - x_1)$, with $x_1 \in \mathbb{R}$, generate the same Pareto outcome.

Next, consider the maximization with respect to (x_1, x_2) of the weighted sum

$$\alpha_1 J_1(x_1, x_2) + \alpha_2 J_2(x_1, x_2) = \alpha_1(x_1 + x_2) - \alpha_2(x_1 + x_2)^3,$$

where $\alpha_i \geq 0$ and $(\alpha_1, \alpha_2) \neq (0, 0)$.

If $\alpha_2 \neq 0$, this weighted average can be made arbitrarily large by choosing $x_1 + x_2$ sufficiently negative. If instead $\alpha_2 = 0$ (and hence $\alpha_1 > 0$), the objective can be made arbitrarily large by choosing $x_1 + x_2$ sufficiently large. Therefore, for any choice of the weights α_i , the weighted-sum objective does not attain a maximum.

EXAMPLE 6.16 We continue Example 6.15 by examining the necessary conditions implied by Theorem 6.11. According to this theorem, any solution of the equation

$$\alpha_1 - 3\alpha_2(x_1 + x_2)^2 = 0,$$

constitutes a candidate Pareto efficient solution. Since the coefficients α_i may take any nonnegative values, it follows that every pair $(x_1, x_2) \in \mathbb{R}^2$ qualifies as a potential Pareto solution. Indeed, for any given $(\bar{x}_1, \bar{x}_2)$ one can choose, for example, $\alpha_2 = 1$, and then select a nonnegative α_1 such that

$$(\bar{x}_1 + \bar{x}_2)^2 = \frac{\alpha_1}{3\alpha_2}$$

is satisfied.

In the literature, there also exist second-order conditions that are sufficient to ensure that a set of instruments satisfying the first-order conditions of Theorem 6.11 corresponds to a Pareto-optimal solution. A detailed treatment of these conditions, however, falls beyond the scope of this book. Interested readers are referred to, for example, [24].

6.1.3 An Algorithm to find Pareto Solutions

Determining the Pareto frontier is generally not an easy task. In the following, we outline an algorithm, derived from the previous two subsections, that illustrates a general approach to finding this frontier. We conclude this section by showing how, in numerical methods, the derivatives of the relevant objective functions can be employed to identify Pareto-optimal solutions.

Based on Lemma 6.10 and Theorem 6.11, the following algorithm can be used to compute the Pareto frontier. The algorithm is then illustrated through two examples.

ALGORITHM 6.17 Assume J_i , $i = 1, \dots, N$, g_i , $i = 1, \dots, k$, and h_i , $i = 1, \dots, m$, are continuous differentiable. Let

$$L := \sum_{i=1}^N \alpha_i J_i(x) - \sum_{i=1}^k \lambda_i (g_i(x) - b_i) - \sum_{i=1}^m \mu_i (h_i(x) - c_i).$$

Then the Pareto frontier can be found as outlined in the steps below.

Step 1 Determine all solutions $(x^*, \lambda_1, \dots, \mu_m)$ with $x^* \in \Gamma$, $\alpha_i \geq 0$, $i = 1, \dots, N$, $\lambda_i \geq 0$, $i = 1, \dots, k$, and μ_i , $i = 1, \dots, m$, that satisfy the set of (in)equalities

1. $DL_x(x^*, \lambda_1, \dots, \mu_m) = [0 \ \dots \ 0]$.
2. $\lambda_i (g_i(x^*) - b_i) = 0$, $i = 1, \dots, k$.
3. $g_i(x^*) \leq b_i$, $i = 1, \dots, k$ and $h_i(x^*) = c_i$, $i = 1, \dots, m$.
4. $(\alpha_1, \dots, \alpha_N, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m) \neq (0, \dots, 0)$.

Let $\Delta := \{x^*\}$. Then all Pareto efficient solutions are contained in Δ .

Step 2 For every candidate solution $\bar{x} \in \Delta$ for which the associated α found in Step 1 satisfies $\alpha_i > 0$, $i = 1, \dots, N$, verify whether $\bar{x}$ maximizes $\sum_{i=1}^N \alpha_i J_i(x)$. If this is the case, add $\bar{x}$ to the set of all Pareto efficient locations P .

Step 3 For every candidate solution $\bar{x} \in \Delta \setminus P$ determine the sets

$$\Gamma_i := \{x \in \Gamma \mid J_j(x) \geq J_j(\bar{x}), j = 1, \dots, N, j \neq i\}, \quad \text{for } i = 1, \dots, N.$$

Verify whether $J_i(\bar{x}) \geq J_i(x)$ for all $x \in \Gamma_i$, for $i = 1, \dots, N$. If this property holds, $\bar{x}$ is Pareto efficient and add this solution to the set P .

Step 4 The Pareto frontier is obtained by calculating for all $x \in P$, $(J_1(x), \dots, J_N(x))$.

REMARK 6.18 If, in Algorithm 6.17, the feasible set Γ is compact, and the vector α found in Step 1 satisfies $\alpha_i > 0$ with a corresponding unique solution x^* , then x^* is Pareto efficient. This follows because the associated constrained optimization problem has a maximum by Weierstrass' theorem. Since x^* also satisfies the corresponding first-order conditions, its Pareto efficiency then follows directly from Theorem 6.6.

EXAMPLE 6.19 (*cooperative profit maximization*)

Consider two manufacturers producing the same product for the market. The market price is given by $p := 10 - x_1 - x_2$, where x_i denotes the quantity produced by manufacturer i ($i = 1, 2$). The production costs of manufacturers are $x_1 + 2$ for manufacturer 1 and $2x_2 + 1$ for manufacturer 2.

The manufacturers aim to identify all possible combinations of individual (positive) profits by coordinating their total production.

Thus, the problem is to determine the Pareto frontier of the two-player game in which the profit π_i of player i is defined as $\pi_1 = p * x_1 - (x_1 + 2)$ for player 1 and $\pi_2 = p * x_2 - (2x_2 + 1)$ for player 2, where $x_i \geq 0$ for $i = 1, 2$. Only combinations for which both players obtain positive profits are considered.

Following Algorithm 6.17, we proceed with the following steps, using

$$L(x, \alpha, \lambda) := \alpha_1 \pi_1 + \alpha_2 \pi_2 + \lambda_1 x_1 + \lambda_2 x_2.$$

Step 1. There exist $\alpha_i \geq 0$ and $\lambda_i \geq 0$ such that any Pareto efficient solution $(\bar{x}_1, \bar{x}_2)$ satisfies:

1. $\begin{bmatrix} \bar{p} - \bar{x}_1 - 1 & -\bar{x}_2 \\ -\bar{x}_1 & \bar{p} - \bar{x}_2 - 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $\bar{p} = 10 - \bar{x}_1 - \bar{x}_2$.
2. $\lambda_i \bar{x}_i = 0$, $i = 1, 2$; 3. $\bar{x}_i \geq 0$; 4. $[\alpha_1 \ \alpha_2 \ \lambda_1 \ \lambda_2] \neq 0$.

Notice that in case $\bar{x}_i = 0$, profit for player i is negative. Therefore we can restrict the search for solutions of these conditions to the case $\bar{x}_i > 0$, $i = 1, 2$. Then, see Remark 6.14 item 2, the determinant of matrix $G := \begin{bmatrix} \bar{p} - \bar{x}_1 - 1 & -\bar{x}_2 \\ -\bar{x}_1 & \bar{p} - \bar{x}_2 - 2 \end{bmatrix}$ must equal zero. That is:

$$2(\bar{x}_2^2 + (2\bar{x}_1 - 13)\bar{x}_2 + \bar{x}_1^2 - \frac{25}{2}\bar{x}_1 + 36) = 0.$$

Under the assumption that $(\bar{x}_1, \bar{x}_2)$ satisfies above equality it follows from $G\alpha = 0$ that $\alpha_2 = \frac{\bar{p} - \bar{x}_1 - 1}{\bar{x}_2} \alpha_1$. Since $\alpha_i \geq 0$, we conclude that necessarily $(\bar{x}_1, \bar{x}_2)$ must additionally satisfy $\bar{p} - \bar{x}_1 - 1 \geq 0$. In fact this inequality must hold strict as $G\alpha = 0$ and condition 4 above must hold. Furthermore, one easily verifies from $G\alpha = 0$, that $\alpha_1 \neq \alpha_2$. Consequently, the set of potential Pareto efficient solutions, Δ , is given by

$$\{(\bar{x}_1, \bar{x}_2) \mid \bar{x}_2^2 + (2\bar{x}_1 - 13)\bar{x}_2 + \bar{x}_1^2 - \frac{25}{2}\bar{x}_1 + 36 = 0,$$

$$\text{where } \bar{x}_i > 0, J_i(\bar{x}_1, \bar{x}_2) \geq 0 \text{ and } \bar{p} - \bar{x}_1 - 1 > 0\}.$$

Step 2. With $\pi_\alpha(x_1, x_2) := \alpha_1 \pi_1 + \alpha_2 \pi_2$, where $\alpha_i > 0$, some elementary calculations show that

its second order derivative is $\bar{H} := H(\pi_\alpha(x_1, x_2)) = \begin{bmatrix} -2\alpha_1 & -\alpha_1 - \alpha_2 \\ -\alpha_1 - \alpha_2 & -2\alpha_2 \end{bmatrix}$. Therefore the

determinant of $-\bar{H} = -(\alpha_1 - \alpha_2)^2$. From this we straightforwardly infer that $\bar{H}$ is strict indefinite if $\alpha_1 \neq \alpha_2$. So, under these conditions any stationary point of $\pi_\alpha(x_1, x_2)$ will be a saddle-point. Therefore this step does not help to reduce the set of potential candidate solutions Δ .

Step 3. With $\Gamma_1 := \{(x_1, x_2) \in \Gamma \mid px_2 - (2x_2 + 1) \geq \beta_2\}$, where $\beta_2 := \bar{p}\bar{x}_2 - (2\bar{x}_2 + 1)$, we consider the maximization of π_1 on this set Γ_1 . Notice this set Γ_1 is compact and π_1 has no stationary point in the interior of Γ_1 . So, the maximum is attained at its boundary. Clearly, this maximum is not attained at either $x_1 = 0$ or $x_2 = 0$. So, the maximum is attained at the boundary $\{(x_1, x_2) \in \Gamma \mid px_2 - (2x_2 + 1) = \beta_2\}$. That is, where $x_1 = 8 - x_2 - \frac{1+\beta_2}{x_2}$, where $x_i > 0$, $i = 1, 2$. With $\pi_1(x_2) = 9x_1(x_2) - x_1^2(x_2) - x_1(x_2)x_2 - 2$, it follows that $\pi_1'(x_2) = (9 - 2x_1 - x_2)x_1' - x_1 = \frac{1}{x_2^3}(-x_2^3 - 7(1 + \beta_2)x_2 + 2(1 + \beta_2)^2)$. Notice $1 + \beta_2 = (\bar{p} - 2)\bar{x}_2$. Substitution of this into $\pi_1'(\bar{x}_2)$ yields

$$\begin{aligned}\pi_1'(\bar{x}_2) &= \frac{1}{\bar{x}_2^3}(-\bar{x}_2^3 - 7\bar{x}_2^2(\bar{p} - 2) + 2\bar{x}_2^2(\bar{p} - 2)^2) \\ &= \frac{\bar{x}_2^2}{\bar{x}_2^3}(-\bar{x}_2 - 7\bar{p} + 14 + 2\bar{p}^2 - 8\bar{p} + 8) \\ &= 0,\end{aligned}$$

where in the last step we used the definition of $p = 10 - x_1 - x_2$ to rewrite the equality $\bar{x}_2^2 + (2\bar{x}_1 - 13)\bar{x}_2 + \bar{x}_1^2 - \frac{25}{2}\bar{x}_1 + 36 = 0$ as a function of x_2 and p , i.e., as $2p^2 - 15p + 22 - \bar{x}_2 = 0$.

Since the roots of $\pi_1'(x_2) = 0$ are determined by a third order polynomial from which it is easily verified that it has precisely one root, a simple inspection of this derivative shows that $\pi_1(x_2)$ has a global maximum at $\bar{x}_2$.

In a similar way one can show that also $\pi_2(x_1)$ has a global maximum at $\bar{x}_1$ on $\Gamma_2 := \{(x_1, x_2) \in \Gamma \mid px_1 - (x_1 + 2) \geq \beta_1\}$, where $\beta_1 := \bar{p}\bar{x}_1 - (\bar{x}_1 + 2)$.

Therefore all points $(x_1, x_2) \in \Delta$ for which $\pi_i(x_1, x_2) \geq 0$, $i = 1, 2$, are Pareto efficient.

Step 4. Because $\bar{p} - \bar{x}_1 - 1 > 0$ it follows that the set of Pareto efficient solutions, as characterized in Δ , can be parameterized by

$$(x_1, \frac{13 - 2x_1 - \sqrt{25 - 2x_1}}{2}), \text{ where } 0 < x_1 < 4\frac{1}{2}.$$

In Figure 6.4 we plotted the corresponding solutions $(\pi_1(x_1, x_2), \pi_2(x_1, x_2))$ yielding nonnegative profits.

The next Example 6.20 shows that, even when the constraint set is compact and all the conditions of Theorem 6.11 are satisfied, it may still occur that solutions fulfilling the necessary conditions stated in Theorem 6.11, with strictly positive values of α_i , are Pareto efficient but do not maximize any (strict) weighted average of the players' objectives.

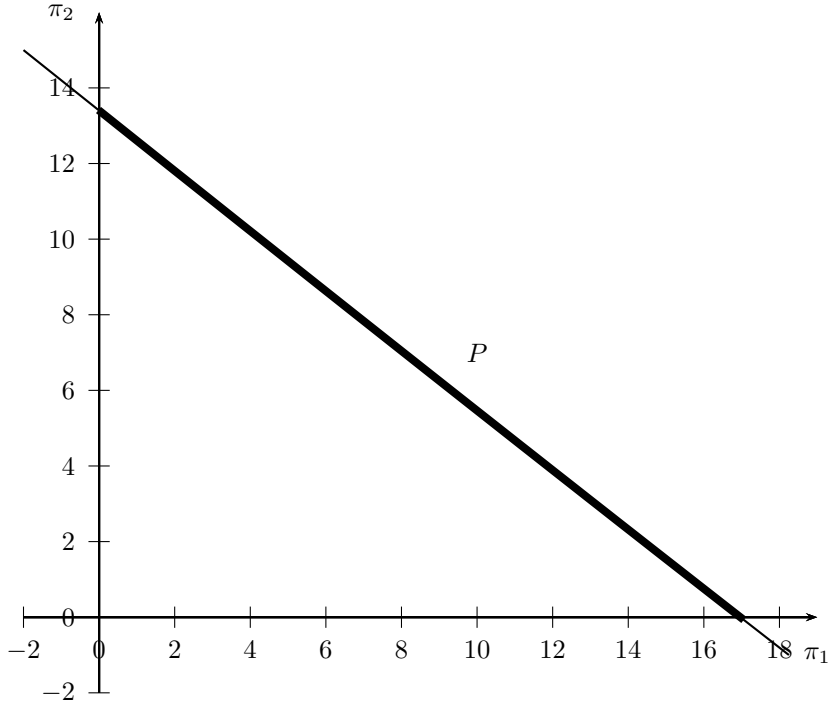


FIGURE 6.4 – The Pareto frontier P (boldfaced part of curve) for Example 6.19.

EXAMPLE 6.20 Consider two players who want to maximize the objectives $J_i = x_i^2$, $i = 1, 2$, over the unit simplex

$$\mathcal{S} = \{(x_1, x_2) \mid x_1 + x_2 \leq 1, x_i \geq 0, i = 1, 2\}.$$

Applying Algorithm 6.17 we introduce the Lagrange function

$$L(x_1, x_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \lambda) := \alpha_1 x_1^2 + \alpha_2 x_2^2 + \lambda_1 x_1 + \lambda_2 x_2 - \lambda(x_1 + x_2 - 1).$$

Step 1. In the first step, Theorem 6.11 yields the following necessary conditions

1. $2\alpha_i x_i + \lambda_i - 2\lambda = 0$, $i = 1, 2$;
2. $\lambda_i x_i = 0$, $i = 1, 2$, and $\lambda(x_1 + x_2 - 1) = 0$;
3. $x_i \geq 0$ $i = 1, 2$, and $x_1 + x_2 \leq 1$;
4. $(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \lambda) \neq 0$, where α_i, λ_i , $i = 1, 2$, and λ are all non-negative.

Some elementary calculations show that the set of solutions to this set of (in)equalities is:

$$\begin{aligned} \Delta := & \{(0, 0, \alpha_1, \alpha_2, 0, 0, 0), (\alpha_1, \alpha_2) \neq 0\}; \\ & (0, 1, \alpha_1, \alpha_2, 2\alpha_2, 0, 2\alpha_2), \text{ where } \alpha_2 > 0; \\ & (0, x_2, \alpha_1, 0, 0, 0, 0), \text{ where } \alpha_1 > 0 \text{ and } 0 < x_2 \leq 1; \\ & (1, 0, \alpha_1, \alpha_2, 0, 2\alpha_1, 2\alpha_1), \text{ where } \alpha_1 > 0; \\ & (x_1, 0, 0, \alpha_2, 0, 0, 0), \text{ where } \alpha_2 > 0 \text{ and } 0 < x_1 \leq 1; \text{ and} \\ & (x_1, 1 - x_1, \alpha_1, \alpha_1 \frac{x_1}{1 - x_1}, 0, 0, 2\alpha_1 x_1), \text{ where } x_1 \in (0, 1) \text{ and } \alpha_1 > 0\}. \end{aligned}$$

That is, all points on the boundary of set $\mathcal{S}$ qualify as Pareto solutions.

Step 2. In Step 2 of the algorithm, we consider the maximization of $J_\alpha := \alpha_1 x_1^2 + \alpha_2 x_2^2$ over $\mathcal{S}$, with $\alpha_i > 0$, $i = 1, 2$. Clearly, for any choice of (α_1, α_2) , J_α attains its global maximum at either $(1, 0)$ or $(0, 1)$. Hence, both points are Pareto efficient solutions.

Step 3. In Step 3 of the algorithm, we examine whether any additional points on the boundary of the set $\mathcal{S}$ are Pareto efficient.

Consider first a point $\bar{x} := (\bar{x}_1, 0)$, with $\bar{x}_1 < 1$. In this case, $J_1(\bar{x}) < J_1(1, 0)$ while $J_2(\bar{x}) = J_2(1, 0)$. Hence, $\bar{x}$ is not Pareto efficient.

Similarly, for a point $\bar{x} := (0, \bar{x}_2)$ with $\bar{x}_2 < 1$, we have $J_1(\bar{x}) = J_1(0, 1)$ and $J_2(\bar{x}) < J_2(0, 1)$, and thus these points are neither Pareto efficient.

Finally, consider a point $(\bar{x}, 1 - \bar{x})$, with $\bar{x} \in (0, 1)$, lying on the boundary line segment $x_1 + x_2 = 1$ of $\mathcal{S}$. Define

$$\Gamma_1 = \{(x_1, x_2) \in \mathcal{S} \mid J_2(x_1, x_2) \geq J_2(\bar{x}, 1 - \bar{x}) = \{(x_1, x_2) \in \mathcal{S} \mid x_2 \geq 1 - \bar{x}\}.$$

Since $x_1 + x_2 \leq 1$, it follows that $x_1 \leq 1 - x_2 \leq \bar{x}$. Hence,

$$J_1(x_1, x_2) = x_1^2 \leq \bar{x}_1^2 \text{ for all } (x_1, x_2) \in \Gamma_1,$$

and thus J_1 attains a global maximum at $(\bar{x}, 1 - \bar{x})$ on Γ_1 .

An analogous argument shows that J_2 attains a global maximum at $(\bar{x}, 1 - \bar{x})$ on

$$\Gamma_2 = \{(x_1, x_2) \in \mathcal{S} \mid J_1(x_1, x_2) \geq J_1(\bar{x}, 1 - \bar{x}) = \{(x_1, x_2) \in \mathcal{S} \mid x_1 \geq \bar{x}\}.$$

Step 4. Consequently, by Lemma 6.10, all points on the line segment $x_1 + x_2 = 1$, $x_i > 0$, are Pareto efficient.

Observe that for any point $(\bar{x}, 1 - \bar{x})$, with $\bar{x} \in (0, 1)$, on the line segment $x_1 + x_2 = 1$, and for $\alpha_1 > 0$ and $\alpha_2 = \frac{\bar{x}}{1 - \bar{x}} \alpha_1$, the weighted objective

$$J_\alpha = \alpha_1 J_1 + \alpha_2 J_2$$

attains its global maximum at $(1, 0)$ if $\bar{x} \leq \frac{1}{2}$, and at $(0, 1)$ if $\bar{x} \geq \frac{1}{2}$.

Furthermore, as shown above, the point $(\bar{x}, 1 - \bar{x})$ also satisfies the first-order conditions implied by the associated Lagrange function

$$L(x_1, x_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \lambda),$$

for suitable choices of λ_i and λ on S . Thus, although $(\bar{x}, 1 - \bar{x})$ does not maximize L on S for this particular choice of α_i , it is nevertheless Pareto efficient.

We conclude this subsection with an observation that is particularly useful when pursuing numerical methods for determining the Pareto frontier. A further necessary condition for $(J_1(\hat{x}), \dots, J_N(\hat{x}))$ to be Pareto optimal is that there exists no direction at $\hat{x}$ along which all objective functions J_i increase strictly. In other words, there should be no direction u at $\hat{x}$ such that the directional derivative of every objective function J_i in the direction u is strictly positive.

Equivalently, for any direction u , it can not be the case that $D_u J_i(\hat{x}) > 0$ for all i simultaneously.

Recall that if the functions J_i are differentiable, then $D_u J_i(\hat{x}) = DJ_i(\hat{x})u$. Hence, if $\hat{x}$ is Pareto efficient, there exists no direction u such that the vector

$$\begin{bmatrix} DJ_1(\hat{x}) \\ \vdots \\ DJ_N(\hat{x}) \end{bmatrix} u > 0^{21}. \quad (6.3)$$

Points $\tilde{x}$ for which no direction u exists such that $DJ(\tilde{x})u > 0$ are called *Pareto critical points*. If a point x is not Pareto critical, then there exists a direction u along which all objective functions J_i increase from x . Consequently, one can move in that direction to find a new point where all values of J_i are larger.

This property is exploited in numerical "steepest descent" algorithms to determine the Pareto frontier by iteratively repeating this step (see, e.g., [10]).

EXERCISE 6.4 Find the Pareto frontier in the next cooperative optimization problems.

a. (*optimal production planning*)

i) A baker can produce both bread and cookies. He has just one oven and can bake either twenty breads or two hundred cookies. The production time of twenty breads is 1 hour and of two hundred cookies 10 minutes. Assume he can use the oven for 8 hours. Calculate the number of bread and cookies he can produce.

ii) Assume the price of one bread is 2 euro and that of 20 cookies is 1 euro. Calculate how many bread and cookies he should produce to maximize his revenues.

b. (*sharing limited production resources*)

Consider two brothers who own a sawing mill. One of the brothers produces shelves which he sells to industry. The other one uses the mill to produce material for his furniture company. The time to

produce one bundle, X , of shelves is 1 hour. The time to produce the specific material needed for the furniture, Y , is 2 hours. The mill can be used 16 hours a day. Assume the price of X is 20 euro and the price of Y is 50 euro. Calculate the number of shelves and specific material the brothers can optimally produce. What will be the optimal production if they decide to use the mill in such a way that their profits are the same.

c. (*fiscal coordination*)

Consider optimal fiscal policies in a large country 1 and a small country 2. Assume that output, y_i , in each country is affected by fiscal instruments, f_i , used by both countries (more precisely, f_i models the deviation of fiscal policies from their current equilibrium value). Let $y_1 = 4\bar{y} + 0.9f_1 + 0.1f_2$ and $y_2 = \bar{y} + 0.8f_2 + 0.2f_1$, where $\bar{y}$ is some given number. Assume both countries are in a recession and like to raise output to $5\bar{y}$ and $1.2\bar{y}$ respectively. This, of course, using as less as possible fiscal instruments. In country 1 this is formulated as the objective to minimize $(y_1 - 5\bar{y})^2 + f_1^2$ and in country 2 as the objective to minimize $(y_2 - 1.2\bar{y})^2 + f_2^2$. Find the Pareto frontier for this problem if both countries coordinate fiscal policies and $\bar{y} = 1$. What will be the outcome if both countries decide to minimize the sum of their objectives in that case (the so-called *social outcome*)?

d. (*sharing the cakes*)

Consider two consumers who like to spend (a part of) their income on the consumption of two goods. Their utility from consumption of an amount of x from good 1 and y from good 2 is $U_1(x, y) = 2\ln(x + 1) + \ln(y + 1)$ and $U_2(x, y) = 2\ln(x + 1) + \frac{4}{3}\ln(y + 1)$, whereas their budget constraints are $4x + 2y \leq 2$ and $4x + 2y \leq 4$, respectively.

Determine the solution(s) to this cooperative problem in case the maximum available amount of the goods is 1. The same question if this amount is $\frac{1}{2}$ for both goods.

6.2 Non-cooperative Nash Solution

In this section, we consider the non-cooperative case. Following the approach of the previous section, we study a setting with multiple agents, each of whom has a single objective and one or more instruments at his (her) disposal to influence the economy. Moreover, the variables that each agent seeks to control are also affected by the instruments chosen by the other agents.

In contrast to the previous section, we assume here that the agents do not cooperate in pursuing their objectives. Instead, each agent seeks an action that guarantees the best possible outcome for himself.

EXAMPLE 6.21 (*Cournot competition/competition by setting output quantities*)

Consider two farmers who wish to sell their bananas. The market price p they receive depends on the total quantity of bananas (in kilograms), x , supplied to the market. Assume that the price is given by $p = 100 - x$. The cost function of farmer 1 is $C_1(x_1) = 30x_1 + 150$, and the cost function of farmer 2 is $C_2(x_2) = 35x_2 + 50$.

The profit function of farmer i , $i = 1, 2$, is therefore given by

$$\begin{aligned}\pi_1(x_1, x_2) &= px_1 - C_1(x_1) = (100 - x_1 - x_2)x_1 - (30x_1 + 150), \\ \pi_2(x_1, x_2) &= px_2 - C_2(x_2) = (100 - x_1 - x_2)x_2 - (35x_2 + 50),\end{aligned}$$

respectively.

Clearly, in this example, the profit of one farmer depends on the quantity of bananas supplied to the market by the other farmer.

Without any further assumptions, it is not possible to determine the price the farmers will receive.

Suppose that on day 1 farmer 1 goes to the market with $\bar{x}_1 = 45$ kilograms of bananas. Farmer 2 observes farmer 1 and then chooses the quantity of bananas to supply to the market so as to maximize his own profit. Differentiating $\pi_2(45, x_2)$ with respect to x_2 yields

$$\pi_2' = 65 - \bar{x}_1 - 2x_2.$$

Note that the second-order derivative of π_2 with respect to x_2 is $-2 < 0$. Hence, the stationary point of π_2 corresponds to a maximum. Therefore, given that farmer 1 supplies an amount of $\bar{x}_1$ to the market, the profit-maximizing quantity of bananas for farmer 2 is:

$$x_2 = \frac{1}{2}(65 - \bar{x}_1). \quad (6.4)$$

Thus, farmer 2 chooses to sell $\bar{x}_2 = 10$ kilograms of bananas that day.

Observing what happened the previous day, farmer 2 decides to go to the market with 10 kilograms again the next day. However, this time farmer 1 notices him on his way to the market and chooses to bring the quantity of bananas that maximizes his own profit. Differentiating $\pi_1(x_1, 10)$ with respect to x_1 gives

$$\pi_1' = 70 - 2x_1 - \bar{x}_2.$$

Note that $\pi_1'' < 0$. Thus, in this case as well, the stationary point of π_1 corresponds to a maximum. Therefore, given that farmer 2 brings $\bar{x}_2 = 10$ kilograms to the market, the profit-maximizing quantity of bananas for farmer 1 is

$$x_1 = \frac{1}{2}(70 - \bar{x}_2). \quad (6.5)$$

Consequently, farmer 1 chooses to sell $\bar{x}_1 = 30$ kilograms that day, increasing his profit from 525 the previous day to 750.

The next day, this process repeats. Farmer 1, encouraged by his large profit increase the previous day, decides to go to the market with 30 kilograms of bananas again. However, farmer 2 notices this and

chooses, in order to maximize his own profit that day, to bring

$$x_2 = \frac{1}{2}(65 - 30) = 17.5$$

kilograms to the market (see (6.4)).

This process is illustrated in Figure 6.5. If it repeats daily, the total quantity of bananas sold on the market will eventually be determined by the condition that neither farmer has an incentive to deviate from the quantity they bring, given the quantity supplied by the other farmer.

Since farmer 1 adjusts his supply according to (6.5) and farmer 2 adjusts according to (6.4), the quantities $(\bar{x}_1, \bar{x}_2)$ that both farmers bring to the market, from which neither will want to deviate, correspond to the intersection of these two reaction curves. In other words, $(\bar{x}_1, \bar{x}_2)$ must satisfy both

$$\bar{x}_2 = \frac{1}{2}(65 - \bar{x}_1) \text{ and } \bar{x}_1 = \frac{1}{2}(70 - \bar{x}_2).$$

It is straightforward to verify that these equations have a unique solution: $(\bar{x}_1, \bar{x}_2) = (25, 20)$.

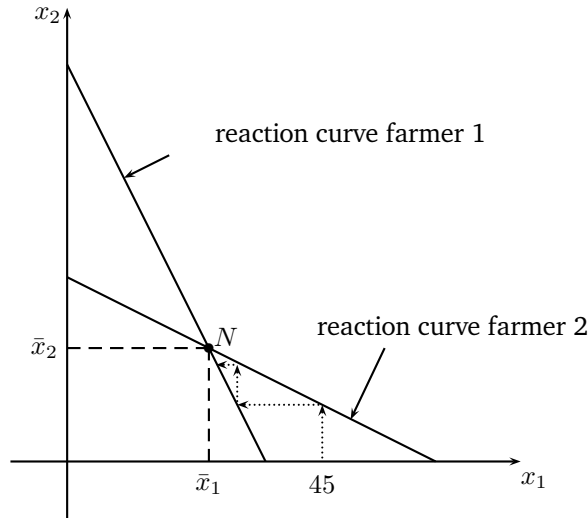


FIGURE 6.5 – Convergence reactions towards Nash equilibrium in Example 6.21.

Example 6.21 motivates the introduction of the notion of a reaction curve, also known in the literature as the *best response set* or *best reply set*. Following the framework of Section 1, let $x_i \in \mathbb{R}^{m_i}$ denote the instrument(s) available to agent i for $i = 1, \dots, N$, and let Γ represent the set of feasible allocations for all instruments $x := (x_1, \dots, x_N)$.

Here, Γ is again described by a set of (in)equalities

$$\Gamma := \{x \in \mathbb{R}^m \mid g_i(x) \leq b_i, i = 1, \dots, k, \text{ and } h_j(x) = c_j, j = 1, \dots, n\}, \text{ with } n = \sum_{i=1}^N m_i. \quad (6.6)$$

Let $J_i(x)$ denote the objective that agent i seeks to maximize, for $i = 1, \dots, N$. Furthermore, let x_{-i} denote the vector x with the i^{th} entry removed.

DEFINITION 6.22 $\hat{x}_i$ is called the *optimal reaction* of player i against the set of instruments $\bar{x}_{-i}$, if

$$\hat{x}_i = \arg \max_{x \in \Gamma_i(\bar{x}_{-i})} J_i(x).$$

Here,

$$\Gamma_i(\bar{x}_{-i}) = \{x \in \Gamma \mid x_j = \bar{x}_j, j \neq i\}.$$

The *reaction curve* of player i , is the set of points $\hat{x}_i(x_{-i})$, where x_{-i} ranges over all instruments for which $\Gamma_i(x_{-i}) \neq \emptyset$.

Thus, the reaction curve of player i consists of all optimal responses of player i to the set of feasible instruments. Note that it is possible that, for some feasible instruments, an optimal response may not exist, or that multiple optimal responses may exist. Conditions that guarantee the reaction curve is a well-behaved, differentiable function can be established, for example, using the Implicit Function Theorem (see Theorem 3.26).

LEMMA 6.23 Assume that $J_i(x) \in C^2$, and at $x = \hat{x}$ we have

$$D_{x_i} J_i(x) = 0 \text{ and } D_{x_i x_i} J_i(x) < 0.$$

Then, in an open neighborhood around $x_i = \bar{x}_i$, the reaction curve $\hat{x}_i(x_{-i})$ exists and is a differentiable function. Moreover, its derivative is given by $D\hat{x}_i(x_{-i}) = -[D_{x_i x_i} J_i(x)]^{-1} D_{x_{-i}}(D_{x_i} J_i(x))$.

EXAMPLE 6.24 (Bertrand competition/competition by setting price levels)

Consider two farmers who sell bananas on a market, where farmer i sets a price p_i . The demand for farmer i 's bananas, $D_i(p_1, p_2)$, is given by

$$D_1(p_1, p_2) = 100 - \frac{5}{4}p_1 + p_2 \text{ and } D_2(p_1, p_2) = 80 - p_1 + p_2.$$

We assume that both farmers supply enough bananas to meet the demand completely, so that the quantity sold by farmer i , x_i , equals D_i .

The cost function of farmer 1 is $C_1(x_1) = 30x_1 + 120$, and the cost function of farmer 2 is $C_2(x_2) = 35x_2 + 5$.

Therefore, the profit function of farmer i , $i = 1, 2$, is

$$\begin{aligned}\pi_1(p_1, p_2) &= p_1 x_1 - C_1(x_1) = (100 - \frac{5}{4}p_1 - p_2)(p_1 - 30) - 120 \text{ and} \\ \pi_2(p_1, p_2) &= p_2 x_2 - C_2(x_2) = (80 - p_1 - p_2)(p_2 - 35) - 5.\end{aligned}$$

The reaction curves of the two farmers are

$$\hat{p}_1(p_2) = 55 - \frac{2}{5}p_2 \text{ and } \hat{p}_2(p_1) = \frac{1}{2}(115 - p_1),$$

for farmers 1 and 2, respectively.

If the farmers sequentially announce the price that maximizes their profit, given the price announced by the other, the resulting equilibrium prices are $p_1 = 40$ and $p_2 = 37.5$.

The pricing process is illustrated in Figure 6.6, starting from an initial price of $p_2 = 50$ for farmer 2.

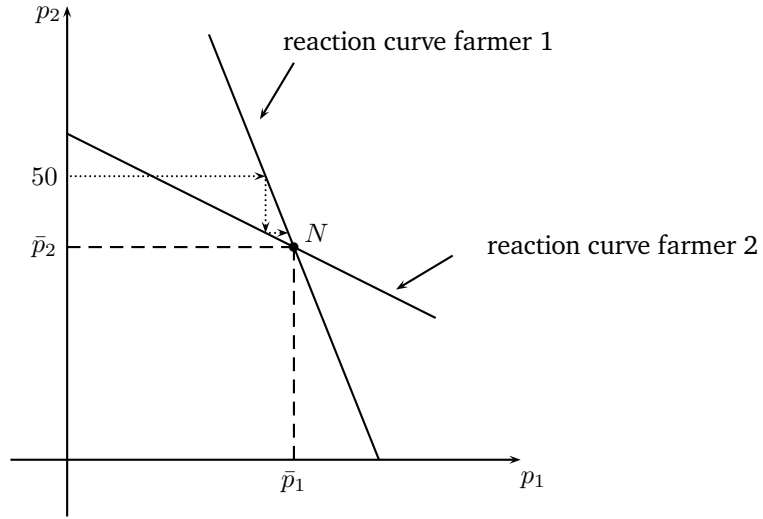


FIGURE 6.6 – Convergence reactions pricing scheme towards Nash equilibrium in Example 6.24.

As Examples 6.21 and 6.24 show, the final equilibrium is found as the intersection of the reaction curves - that is, at a point where no player can improve their outcome by unilaterally deviating. Such a point is called a Nash equilibrium. To formalize this concept, we introduce the notation $\hat{x}_{\setminus i}$, which denotes the vector $\hat{x}$ with its i^{th} entry replaced by an arbitrary element x_i .

DEFINITION 6.25 $\hat{x} \in \Gamma$ is called *Nash efficient* if for all $i = 1, \dots, N$, the following inequality holds

$$J_i(\hat{x}) \geq J_i(\hat{x}_{\setminus i}) \text{ for all } \hat{x}_{\setminus i} \in \Gamma_i(\hat{x}_{-i}).$$

The corresponding point $(J_1(\hat{x}), \dots, J_N(\hat{x})) \in \mathbb{R}^N$ is called a *Nash equilibrium*.

Rephrasing the above, Definition 6.25 states that if there exists a point $\hat{x} \in \Gamma$ such that, for each player i , the entry $\hat{x}_i$ maximizes $J_i(\hat{x}_{\setminus i})$, for all $\hat{x}_{\setminus i} \in \Gamma$, then $\hat{x}$ is called a Nash equilibrium.

Exercise 6.15 illustrates that a game can have no, one, a finite number, or even an infinite number of Nash equilibria.

Theorems 6.26 and 6.30 provide some sufficient conditions for the existence of a Nash equilibrium. Theorem 6.26 is essentially a direct consequence of Brouwer's Fixed Point Theorem (see Exercise 2.57). Under the assumption that each player's optimal reaction is always achieved at a unique point, and that the resulting reaction curves are continuous on their convex domains, the game is guaranteed to have a Nash equilibrium. The reader is invited to prove the details of this result in Exercise 6.29.

THEOREM 6.26 *Let $\Gamma \subset \mathbb{R}^m$ be a convex and compact set. Assume player i has instrument x_i at his disposal to maximize $J_i(x)$, $i = 1, \dots, N$, where $x = (x_1, \dots, x_N) \in \Gamma$.*

Then for all $x \in \Gamma$, the action set $\Gamma_i(x_{-i})$ is a nonempty set that is both convex and compact too.

Furthermore the game has at least one Nash solution if for every player i the reaction curve of player i , $\hat{x}_i(x_{-i})$, where $x_{-i} \in \Gamma_i(x_{-i})$, is a continuous function.

EXAMPLE 6.27 Consider $\Gamma = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Assume player 1 wants to maximize $J_1 = -(x - y)^2$ w.r.t. x and player 2 wants to maximize $J_2 = -(y - x - \frac{1}{2})^2$ w.r.t. y .

Clearly, Γ is a convex and compact set in $\mathbb{R}^2$.

For a fixed $y \in [0, 1]$ one easily verifies that the optimal reaction of player 1 is given by

$$\hat{x}_1(y) = \begin{cases} -\sqrt{1 - y^2}, & \text{if } -1 \leq y \leq -\sqrt{\frac{1}{2}}; \\ y, & \text{if } -\sqrt{\frac{1}{2}} \leq y \leq \sqrt{\frac{1}{2}}; \\ \sqrt{1 - y^2}, & \text{if } \sqrt{\frac{1}{2}} \leq y \leq 1. \end{cases}$$

Whereas for a fixed $x \in [0, 1]$ the optimal reaction of player 2 is given by

$$\hat{y}_2(x) = \begin{cases} -\sqrt{1 - x^2}, & \text{if } -1 \leq x \leq \frac{-1 - \sqrt{7}}{4}; \\ x + \frac{1}{2}, & \text{if } \frac{-1 - \sqrt{7}}{4} \leq x \leq \frac{-1 + \sqrt{7}}{4}; \\ \sqrt{1 - x^2}, & \text{if } \frac{-1 + \sqrt{7}}{4} \leq x \leq 1. \end{cases}$$

Clearly both $\hat{x}_1(y)$ and $\hat{y}_2(x)$ are continuous functions on their (convex) domain $[0, 1]$. So, by Theorem 6.26, this game has a Nash solution.

From Definition 6.25, it follows directly that if $\hat{x}$ is a stationary point for all objective functions, and if each function $J_i(\hat{x}_{\setminus i})$, with $\hat{x}_{\setminus i} \in \Gamma_i(\hat{x}_{-i})$, attains a global maximum at $x_i = \hat{x}_i$, then $\hat{x}$ is a Nash efficient solution. A special case of this observation leads to the following result.

THEOREM 6.28 Let $J_i(x) \in C^2$. Assume $\hat{x}$ is an interior point of Γ such that $D_{x_i} J_i(\hat{x}) = 0$. If, for each $i = 1, \dots, N$, the set $\Gamma_i(\hat{x}_{-i})$ is a connected set and

$$D_{x_i x_i} J_i(\hat{x}_{\setminus i}) \leq 0, \text{ for all } \hat{x}_{\setminus i} \in \Gamma_i(\hat{x}_{-i}),$$

then $\hat{x}$ is a Nash efficient solution.

From this Theorem, we obtain immediately the following result in case of an unconstrained domain.

COROLLARY 6.29 Let $\Gamma = \mathbb{R}^m$ and $J_i(x) \in C^2$. Then a necessary condition for $\hat{x}$ to be Nash efficient is that

$$D_{x_i} J_i(\hat{x}) = 0.$$

Moreover, if

$$D_{x_i x_i} J_i(\hat{x}_{\setminus i}) \leq 0 \text{ for all } \hat{x}_{\setminus i} = (\hat{x}_1, \dots, x_i, \dots, \hat{x}_N) \text{ with } x_i \in \mathbb{R}^{m_i}, i = 1, \dots, N,$$

then $\hat{x}$ is a Nash efficient solution.

The following Theorem 6.30 gives a sufficient condition under which the reaction curves of all players are continuous within the framework of Theorem 6.26. Consequently, it also provides a sufficient condition for the existence of a Nash equilibrium. The reader is invited to prove this result in Exercise 6.30.

THEOREM 6.30 Let Γ be a compact and convex set. Assume that the continuous objective function $J_i(x)$, which player i seeks to maximize, is strictly concave in x_i for all $x \in \Gamma_i(x_{-i})$, for $i = 1, \dots, N$. Then there exists $\hat{x} \in \Gamma$ at which a Nash equilibrium is attained.

EXAMPLE 6.31 Reconsider Example 6.21 on $\Gamma = [0, 1000] \times [0, 1500]$. In this example, it is clear that for a fixed $\bar{x}_2 \in [0, 1500]$,

$$\pi_1(x_1, \bar{x}_2) = (100 - x_1 - \bar{x}_2)x_1 - (30x_1 + 150)$$

is a strictly concave function of x_1 on $[0, 1000]$, and for fixed $\bar{x}_1 \in [0, 1000]$,

$$\pi_2(\bar{x}_1, x_2) = (100 - \bar{x}_1 - x_2)x_2 - (35x_2 + 50)$$

is a strictly concave function of x_2 on $[0, 1500]$.

It is easy to verify that all other assumptions of Theorem 6.30 are also satisfied. Therefore, by Theorem 6.30, this game must have a Nash equilibrium on Γ .

REMARK 6.32 In Example 6.21, we found that the Nash efficient instruments are $(\hat{x}_1, \hat{x}_2) = (25, 20)$, which correspond to the Nash equilibrium $(\hat{\pi}_1, \hat{\pi}_2) = (475, 350)$. Since the instruments in this case are quantities, this type of equilibrium is often referred in economics as a *Cournot(-Nash) equilibrium*, after Cournot, who studied this form of equilibrium in 1838, whereas Nash later generalized the concept in 1953.

In Example 6.24, the Nash efficient instruments are $(\hat{p}_1, \hat{p}_2) = (40, 37.5)$, leading to the Nash equilibrium $(\hat{\pi}_1, \hat{\pi}_2) = (5, 1.25)$. Since the instruments are here prices, this equilibrium is often called a *Bertrand(-Nash) equilibrium*, named after Bertrand, who studied this type of equilibrium in 1883.

EXERCISE 6.5 Reconsider Example 6.24.

- Use Theorem 6.30 to show that this problem has a Nash equilibrium on every set $\Gamma := [0, a] \times [0, b]$, where a and b are some positive constants.
- Plot the reaction functions and next calculate the Nash equilibrium if $a = b = 38$.

From an algorithmic perspective, it is straightforward to determine the Nash equilibria of a game. First, one should identify the best-response function for each player. Then, one should look for the points where all these best-response functions intersect. Conceptually, however, the emergence of Nash equilibria is likely most realistic in a dynamic setting, as illustrated by the examples of Cournot and Bertrand competition discussed above. In a static framework, it is assumed that all agents have complete knowledge of the objectives and responses of all other agents - a condition that, in most cases, is not realistic.

6.3 Intermediate and Hierarchical Solutions

In the previous two sections, we presented a solution to the multi-agent problem for the cases in which either all agents cooperate or none of them do. An intermediate scenario to consider is one in which some agents form a coalition that competes against other agents or coalitions. To analyze such intermediate cases, one can combine the methods developed in the previous sections to create an appropriate problem framework. A comprehensive treatment of this topic is beyond the scope of this book and will therefore not be pursued here. The exercise below is intended to give the reader a sense of how such problems might be approached.

EXERCISE 6.6 (*fiscal coordination*)

Consider three countries where country 1 is in a recession, country 2 is performing well and country 3's economy is overheated. All countries want to use their fiscal policies, f_i (see Exercise 6.4 item c), to adapt output, y_i , in their country. Assume

$$y_1 = 5 + 0.8f_1 + 0.1f_2 + 0.1f_3, \quad y_2 = 10 + 0.1f_1 + 0.8f_2 + 0.1f_3 \quad \text{and} \quad y_3 = 15 + 0.1f_1 + 0.1f_2 + 0.8f_3.$$

The cost functions the different countries like to minimize are given by

$$J_1 = (y_1 - 10)^2 + f_1^2, \quad J_2 = (y_2 - 10)^2 + f_2^2 \quad \text{and} \quad J_3 = (y_3 - 10)^2 + f_3^2,$$

respectively. Assume the first two countries are willing to cooperate in order to tackle the economic problems.

- Formulate this problem as a non-cooperative situation, where country 1 and 2 play in a coalition against country 3. Assume that the two cooperative countries formalize their cooperation in an agreement to minimize a weighted sum of their cost functions, i.e., $\alpha J_1 + (1 - \alpha)J_2$. Determine the reaction functions of the players.
- Determine for fixed α the equations that have to be satisfied by a Nash equilibrium in the game in item a). Calculate, using e.g. a symbolic computerpackage, the Nash efficient equilibrium fiscal policies for this game.
- Use a computerpackage, e.g. Matlab, to plot the Nash efficient fiscal policies in item b) and corresponding output for $\alpha \in (0, 1)$ for country 1 and 2. Comment on the policies as a function of α .
- Similarly, plot the Nash efficient fiscal policy and corresponding output for country 3 as a function of α . Explain again the results.
- Plot the Nash solution $(J_1(\alpha), J_2(\alpha))$ for $\alpha \in (0, 1)$. What is the outcome for $\alpha = 0.5$? Show that for approximately $\alpha = 0.9424$, $J_1(\alpha) = J_2(\alpha)$.

Another solution, conceptually distinct from the cooperative and non-cooperative approaches, arises when we assume a leader-follower structure in the game (for simplicity, consider just two agents). Heinrich von Stackelberg was the first to analyze this type of game in 1934 (see [26]), which is why such games are commonly referred to as Stackelberg games in the literature. In the remainder of this section, we provide a brief introduction to the solution concepts that can be applied in this setting.

In a Stackelberg game, it is assumed that the leader first announces his decision x_L , which is then observed by the other player, the follower. Given this information, the follower chooses his decision x_F by maximizing his objective function for the announced value of x_L . As a result, the follower's optimal response x_F is a function of x_L .

In the original formulation of the Stackelberg game, which we adopt here, the leader is assumed to have complete knowledge of the follower's preferences and strategy and therefore knows the follower's optimal reaction function in advance. When determining his own decision x_L , the leader takes

this reaction function into account and selects x_L as the value that maximizes his objective function $J_L(x_L, x_F(x_L))$.

Clearly, by varying the amount of information available to the leader (or, in a dynamic setting, the information that may become available over time), alternative solution concepts can be considered. However, we do not pursue these possibilities further here. At this point, we merely wish to emphasize the importance of the underlying assumption that, in the present solution concept, the leader is assumed to have complete knowledge of the follower's preferences and strategy. Without this information, the leader would be unable to determine his optimal strategy. Moreover, the leader must also be certain that the follower can observe his action. To illustrate the concept of a Stackelberg game, we revisit Example 6.21.

EXAMPLE 6.33 (*profit maximization in market with two firms*)

In Example 6.21, we assumed that both farmers are aware of the banana price p , which is given by

$$p = 100 - x,$$

where x denotes the total quantity of bananas supplied to the market. Furthermore, we assumed that the cost functions of farmers $i = 1, 2$ are given by $C_1(x_1) = 30x_1 + 150$ and $C_2(x_2) = 35x_2 + 50$, respectively. This leads to the following profit functions for farmers $i = 1, 2$:

$$\begin{aligned}\pi_1(x_1, x_2) &= px_1 - C_1(x_1) = (100 - x_1 - x_2)x_1 - (30x_1 + 150) \text{ and} \\ \pi_2(x_1, x_2) &= px_2 - C_2(x_2) = (100 - x_1 - x_2)x_2 - (35x_2 + 50),\end{aligned}$$

respectively.

Assume that farmer 1 acts as the leader in this market and has full knowledge of the profit function of farmer 2. Moreover, assume that farmer 1 knows that farmer 2 will supply the quantity of bananas that maximizes his profit function π_2 .

As shown in Example 6.21, if farmer 2 observes that farmer 1 supplies $x_1 = \bar{x}_1$ bananas to the market, the optimal response of farmer 2 is given by (see (6.4))

$$x_2 = \frac{1}{2}(65 - \bar{x}_1).$$

Taking this reaction into account, farmer 1 determines his own decision. His resulting profit function is therefore

$$\begin{aligned}\pi_1(x_1, x_2(x_1)) &= px_1 - C_1(x_1) = (100 - x_1 - x_2(x_1))x_1 - (30x_1 + 150) \\ &= (100 - x_1 - \frac{1}{2}(65 - x_1))x_1 - (30x_1 + 150) \\ &= -\frac{1}{2}x_1^2 + 37\frac{1}{2}x_1 - 150.\end{aligned}$$

It is straightforward to verify that farmer 1's profit is maximized when he supplies $x_1 = 37\frac{1}{2}$ kilograms of bananas to the market. Consequently, farmer 1 must arrive at the market early, passing by farm 2, thereby ensuring that farmer 2 observes that he is transporting $x_1 = 37\frac{1}{2}$ kilograms of bananas. Farmer 2 will respond by supplying

$$x_2 = \frac{1}{2}(65 - 37\frac{1}{2}) = 13\frac{3}{4}$$

kilograms. The resulting market price is, therefore, $p = 48\frac{3}{4}$.

This outcome yields profits of $553\frac{1}{8}$ and $139\frac{1}{16}$ for farmers 1 and 2, respectively. Note that this represents a substantial increase in profit for farmer 1 compared to the 475 he would obtain under the Nash equilibrium. In contrast, farmer 2's profit decreases significantly, from 350 to $139\frac{1}{16}$. As a result, total profits under the Stackelberg solution are lower than in the Nash case.

To compare the cooperative, Nash and Stackelberg outcomes for this game, all three solutions are plotted in Figure 6.7. In particular, the figure shows that the non-cooperative Nash equilibrium lies closer to the Pareto frontier than does the Stackelberg solution.

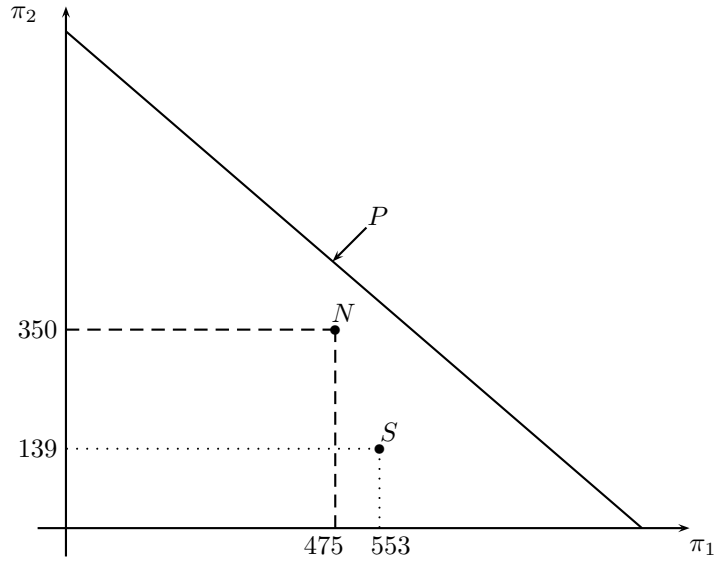


FIGURE 6.7 – The Pareto (P), Nash (N) and Stackelberg (S) solution in Example 6.33.

In the discussion above, it was assumed that the follower's optimal response to every action of the leader is unique. In general, however, this need not be the case. Therefore, before presenting the formal definition of the Stackelberg solution, we first introduce the concept of the best response set.

DEFINITION 6.34 The *best response set* of the follower towards a given x_L is defined as

$$\mathcal{R}(x_L) = \{x_F^* \mid x_F^* \text{ maximizes } J_F(x_L, x_F) \text{ for all } x_F \text{ such that } (x_L, x_F) \in \Gamma\}.$$

DEFINITION 6.35 Assume that for every action of the leader, x_L , the best response set of the follower is unique. Let $x_F(x_L)$ denote the optimal reaction function of the follower and, with some abuse of notation, $x_F^* := x_F(x_L^*)$. Then, $(J_L(x_L^*, x_F^*), J_F(x_L^*, x_F^*))$ is called a *Stackelberg solution* if

$$J_L(x_L^*, x_F(x_L^*)) \geq J_L(x_L, x_F(x_L)) \text{ for all } x_L \text{ such that } (x_L, x_F(x_L)) \in \Gamma.$$

Every action $(x_L^*, x_F(x_L^*))$ that yields a Stackelberg solution is called *Stackelberg efficient*.

Note that in a two-player game, the leader can always enforce a Nash equilibrium by choosing his Nash-equilibrium action. Consequently, if a Nash equilibrium exists, the leader's payoff in the Stackelberg setting is at least as high as the payoff he would obtain in the Nash equilibrium.

EXERCISE 6.7 Reconsider Example 6.24.

- Determine the Stackelberg solution and its corresponding efficient actions if farmer 1 is the leader. Is this a realistic scenario?
- In order to persuade farmer 2 to satisfy the demand of his clients which price should farmer 1 charge at most for his bananas?
- Answer the same question as in item a) if farmer 2 is the leader.
- Plot the Pareto frontier, the Nash solution and the Stackelberg solutions of item b) and c) in one figure. Compare the distances of the last three solutions to the Pareto frontier and conclude.

EXERCISE 6.8 Reconsider the fiscal policy coordination problem, Exercise 6.4 item c.

- Assume country 1 acts as a leader in this fiscal policy game. Determine the corresponding Stackelberg solution. Compare this solution with the Nash solution and conclude.
- Answer the same questions as in item a) but now with country 2 acting as a leader.
- Plot the Stackelberg solution, the Nash solution and the Pareto frontier in one figure. Compare the distances of the different solutions to the Pareto frontier and conclude.

EXERCISE 6.9 Consider two companies L and F who produce the same product and sell their product in the same market. The price they get for their product is $p = 5 - (q_L + q_F)$, where q_i is the amount of product company i supplies to the market, $i = L, F$. The cost function of the companies are $C_L(q_L) = q_L^4 - 8q_L^3 + 21\frac{1}{4}q_L^2 - 20q_L$ and $C_F = q_F + q_F^2$, respectively.

- Determine the profit for company L if it is the leader in this Stackelberg game.
- What will be the Stackelberg efficient action and corresponding profit of company F in this game if company L likes to minimize profit of its competitor?

In general, the set of Stackelberg-efficient actions is not unique. Consequently, several refinements of the Stackelberg solution concept have been proposed in the literature, in situations where the best response set is not unique. Below, we introduce two such refinements: the max-max and the

max–min Stackelberg solutions. Readers interested in further details or alternative refinements are referred to the literature.

The basic idea behind the max–max concept is that the follower acts in a way that benefits the leader by selecting, from his best response set, the action that yields the highest payoff for the leader. The max–min concept, in contrast, is motivated by the desire to guarantee the leader a minimum payoff, regardless of the action chosen by the follower. This concept is particularly relevant when the follower does not have access to the leader's preferences. In that case, irrespective of the follower's specific rational choice, the leader knows the minimum payoff he can secure.

Using the same notation as before, we have:

DEFINITION 6.36 $(J_L(x_L^{**}, x_F^{**}), J_F(x_L^{**}, x_F^{**}))$ is called a

- *max-max Stackelberg solution* if $x_F^{**} \in \mathcal{R}(x_L^{**})$ and x_F^{**} maximizes $J_L(x_L^*, x_F(x_L^*))$ for all $x_F^* \in \mathcal{R}(x_L^{**})$.
- *max-min Stackelberg solution* if $x_F^{**} \in \mathcal{R}(x_L^{**})$ and x_F^{**} minimizes $J_L(x_L^*, x_F(x_L^*))$ for all $x_F^* \in \mathcal{R}(x_L^{**})$.

EXAMPLE 6.37 To illustrate the concepts introduced above, consider the following example. Let $r = x_2 - x_1$, where $x_i \in [0, 1]$ denotes the control variable of player i . Player 1 aims to maximize r , while player 2 seeks to maximize the control effort exerted by player 1.

If player 1 acts as the leader, it is clear that player 2 is indifferent with respect to his choice of action, regardless of the value of x_1 chosen by player 1. That is, the best response set of player 2 is given by

$$\mathcal{R}(x_1) = [0, 1].$$

If player 2 adopts a cooperative stance, he will choose the action that helps player 1 maximize his objective. This leads to the max-max efficient actions $x_1 = 0$ and $x_2 = 1$. The corresponding max-max Stackelberg outcome is $J_1 = 1$ and $J_2 = 0$.

In contrast, the max-min Stackelberg outcome $J_1 = 0$ and $J_2 = 0$ is achieved when both players choose $x_i = 0$.

If player 2 acts as the leader, the best response set of player 1 is given by

$$\mathcal{R}(x_2) = 0,$$

independent of the value of x_2 chosen by player 2.

Note that in this case there is no unique optimal action for player 2. This clearly calls for a refinement of the concepts introduced above; however, such refinements are beyond the scope of this section.

EXERCISE 6.10 (*worst-case growth optimization*)

Consider a farmer growing potatoes. Using artificial manure, f , he can affect the revenue of his crop,

C . This revenue is, however, also subject to weather conditions, w . Assume $C = s_0\sqrt{f} - w$, where $w \in [-1, 1]$. Moreover, assume the farmer gets a price p for his crop potatoes and the cost per unit manure is λ .

- Formulate the above problem as a game problem, where the farmer is the leader and nature is the follower and nature is indifferent in its choice of w .
- Determine the max-min Stackelberg solution and corresponding efficient instruments. Interpret this solution in terms of the price λ of fertilizer.
- Formalize the Stackelberg solution concept that yields the same solution for the farmer as in item b), in case nature is the leader of the game.

EXERCISE 6.11 Consider player L who wants to maximize $J_L := -(x_L + x_F + 1)^2(x_L^2 + 1)^2$ w.r.t. x_L and player F who wants to maximize $J_F := -(x_F - x_L)^2(x_F + x_L)^2$ w.r.t. x_F . Assume player L is the leader in this game.

Determine both the max-max and the max-min Stackelberg solution in this game.

Another interesting extension of the Stackelberg concept that has been studied in the literature (see, for example, [19]) arises when the leader has flexibility in choosing his action set, and this choice influences the set of outcomes available to the follower. In such cases, the leader can manipulate the follower's behavior in a way that is favorable to him by selecting an action set from a collection of possible action sets at his disposal. This line of reasoning leads to the design of *incentive mechanisms*.

Consider, for instance, a bank (the leader) and an investor (the follower). The investor purchases stocks, with the bank acting as an intermediary. Suppose the investor buys stocks worth x_2 euros. The bank may then charge transaction costs of the form $x_1 = \alpha x_2 + \beta$ euros. By appropriately choosing the parameters α and β , the bank can attempt to maximize its profit, taking into account the investor's reaction function. The following Example 6.38 illustrates this idea.

Note that the sequence of actions in this incentive-based Stackelberg game differs from that in the "original" Stackelberg formulation. In the original Stackelberg game, the leader first commits to and implements his action based on his knowledge of the follower's anticipated response - and the follower subsequently implements the action that the leader has accounted for in optimizing his objective. In the incentive Stackelberg game, by contrast, the leader first announces how he will respond to the follower's actions. In doing so, the leader effectively commits to a response function that maximizes his objective. Anticipating this response, the follower then chooses his optimal action. Finally, the leader implements his action in accordance with the previously announced response function and the follower's realized action.

Determining the optimal response function for the leader in this setting is, in general, a challenging problem. We do not pursue this issue further here and refer the interested reader to the aforementioned literature and the references therein.

EXAMPLE 6.38 Consider a game with a leader, L , and follower, F , who want to minimize their cost function

$$J_L(x_L, x_F) = (x_F - 5)^2 + x_L^2 \text{ and } J_F(x_L, x_F) = x_L^2 + x_F^2 - x_L x_F,$$

respectively.

Clearly, the minimum cost for the leader is zero. This value is achieved when $x_F = 5$ and $x_L = 0$. Now, suppose the leader announces that he will choose

$$x_L = \alpha x_F + \beta,$$

whenever the follower selects x_F . The question we address here is whether the leader can choose the parameters α and β in such a way that his cost is zero.

For the outcome $(x_F, x_L) = (5, 0)$ to solve the problem, this point must lie on the leader's announced strategy curve $x_L = \alpha x_F + \beta$. This condition implies that the parameters must satisfy

$$\beta = -5\alpha.$$

The question raised above therefore reduces to determining whether there exists a value of α such that $x_F = 5$ emerges as the minimizer of $J_F(\alpha x_F - 5\alpha, x_F)$. Straightforward differentiation of J_F with respect to x_F shows that its minimum is attained at

$$x_F = \frac{10\alpha^2 - 5\alpha}{2\alpha^2 - 2\alpha + 2}.$$

Simple calculations then reveal that choosing $\alpha = 2$ yields $x_F = 5$.

Thus, by announcing and committing to the strategy $x_L = 2x_F - 10$ the leader ensures that his minimum attainable cost of zero is achieved.

EXERCISE 6.12 Consider a game with a leader, L , and follower, F , who want to minimize their cost function

$$J_L(x_L, x_F) = (x_L - a)^2 + (x_F - b)^2 \text{ and } J_F(x_L, x_F) = (x_L - c)^2 + (x_F - d)^2, \text{ where } a \neq c,$$

respectively. Determine an affine incentive strategy $x_L = \alpha x_F + \beta$ for the leader such that his cost are zero.

Mixed Exercises

EXERCISE 6.13 Assume player 1 wants to maximize $J_1(x_1, x_2) = 1$ w.r.t. x_1 and player 2 wants to maximize $J_2(x_1, x_2) = (x_2 - 1)^2(x_2 - 2)^2$ w.r.t. x_2 , where $(x_1, x_2) \in \Gamma := \{(x_1, x_2) \mid 1 \leq x_1 \leq 2 \text{ and } 0 \leq x_2 \leq 3, i = 1, 2\}$.

- Determine the optimal reaction sets $\hat{x}_i(x_{-i})$, $i = 1, 2$.
- Show that the game has a unique Nash equilibrium, whereas the game has an infinite number of Nash efficient locations.

EXERCISE 6.14 Use Algorithm 6.17 to find the Pareto frontier in the following cases.

- If a player wants to maximize both $J_1(x_1, x_2, x_3) = x_1 - x_2$ and $J_2(x_1, x_2, x_3) = x_2 + x_3$ w.r.t. x_j , $j = 1, 2, 3$, where $(x_1, x_2, x_3) \in \Gamma := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$.
- If a player wants to maximize simultaneously $J_1(x_1, x_2) = x_1(1 - x_1 - x_2) - (x_1 + x_2)$ and $J_2(x_1, x_2) = x_2(1 - 2x_1 - 2x_2) - (x_1 + x_2)$, w.r.t. $x_j \geq 0$, $j = 1, 2$.
- If a player wants to maximize simultaneously $J_1(x_1, x_2) = x_1(5 - x_1 - x_2) - (x_1 + x_2)$ and $J_2(x_1, x_2) = x_2(10 - 2x_1 - 2x_2) - (x_1 + x_2)$, w.r.t. $x_j \geq 0$, $j = 1, 2$.
- If a government wants to spend its budget $b > 0$ on 3 projects with utilities: $U_i(x_i) = \beta_i x_i^2$, where $\beta_i > 0$ and $x_i \geq 0$, $i = 1, 2, 3$.

EXERCISE 6.15 Assume player i wants to maximize $J_i(x_1, x_2)$ w.r.t. x_i , where in items a)-c) below, $(x_1, x_2) \in \Gamma := \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}$. Determine the Nash equilibria in the following games.

- $J_1(x_1, x_2) = x_1$, $J_2(x_1, x_2) = x_2$.
- $J_1(x_1, x_2) = -x_1^2 + (x_2 - \frac{1}{2})^2$, $J_2(x_1, x_2) = x_1^2 - x_2^2$.
- $J_1(x_1, x_2) = -x_1^2$, $J_2(x_1, x_2) = x_2^2$.
- $J_1(x_1, x_2) = -(x_1 - x_2)^2$, $J_2(x_1, x_2) = -(x_1 - x_2 + \frac{1}{2})^2$, where $(x_1, x_2) \in \Gamma := \mathbb{R}^2$.
- $J_1(x_1, x_2) = -\frac{1}{2}x_1^2 + x_1x_2$, $J_2(x_1, x_2) = -\frac{1}{4}x_2^4 + x_1x_2$, where $(x_1, x_2) \in \Gamma := \mathbb{R}^2$.
- $J_1(x_1, x_2) = 2x_1^{\frac{1}{2}}x_2^{\frac{1}{2}} - x_1$, $J_2(x_1, x_2) = 3x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} - 2x_2$, where $x_i > 0$.
- $J_1(x_1, x_2) = (\alpha x_1^{\rho_1} + (1 - \alpha)x_2^{\rho_1})^{\frac{v}{\rho_1}} - \beta_1 x_1$, $J_2(x_1, x_2) = x_1^{1-\rho_2} x_2^{\rho_2} - \beta_2 x_2$, with $v \neq \rho_1$, $v \neq 1$, $\rho_1 < 1$, $\alpha \in (0, 1)$, $\rho_2 \in (0, 1)$, $\beta_2 > 0$, and $x_i > 0$, $i = 1, 2$.
- $J_1(x_1, x_2) = -\frac{1}{3}x_1^3 + \frac{1}{2}(1 + x_2)x_1^2 - x_1x_2$, $J_2(x_1, x_2) = -x_2^2 + 2\alpha x_1x_2 + 2\alpha(1 - \alpha)x_2$, where $x_i \geq 0$ and $\alpha > 0$. Hint: first determine the optimal reaction curves of both players. Next, determine their intersection as a function of α .

EXERCISE 6.16 Reconsider Exercise 6.15. Determine the Pareto efficient solution(s) for items a-f.

EXERCISE 6.17 Reconsider Exercise 6.15. Assume player 1 is the leader in all items. Determine the Stackelberg (efficient) solution(s) for items a-f.

EXERCISE 6.18 Reconsider Exercise 6.15, item d. Assume player 1 is the leader in this game who announces to use an affine incentive strategy w.r.t. the actions played by the follower. Determine an incentive strategy within this class of strategies which will maximize the objective of the leader.

EXERCISE 6.19 Consider player L who wants to maximize $J_L := -(x_L + x_F + 1)^2(x_L^2 + 1)^2$ w.r.t. x_L and player F who wants to optimize $J_F := -(x_F^2 + 1)^2(x_F + x_L)^2$ w.r.t. x_F . Assume player L is the leader in this game.

- Determine the Stackelberg solution in this game if the follower wants to maximize his objective.
- Is it possible to determine a Stackelberg solution in this game if the follower wants to minimize his objective?
- Next consider the case player F is the leader and player L is the follower in the game and player L wants to maximize his objective. Determine the Stackelberg solution in this game.
- Compare the answers in item a. and c. Is there a preference for leadership in this game?
- Show, using Corollary 6.29, that the only Nash efficient candidate solutions are $(x_L, x_F) = (-2, 1)$ and $(x_L, x_F) = (-1, 1)$.
- Show, using the definition, that both strategies $(x_L, x_F) = (-2, 1)$ and $(x_L, x_F) = (-1, 1)$ are not Nash efficient.

Advanced, Deepening and Broadening Exercises

EXERCISE 6.20 Consider a player who wants to maximize simultaneously both $J_1(x) := 2x^2$ and $J_2(x) := -x^2$.

- Use Algorithm 6.17 to find the Pareto frontier.
- Consider the general problem setting from Definition 6.3 to find the Pareto frontier. Show that even in case there exist $\alpha_i > 0$ such that $x^* \in \Gamma$ minimizes $\sum_{i=1}^N J_i(x)$, this doesn't exclude x^* to be Pareto efficient.

EXERCISE 6.21 (*Example where an equilibrium exists that is locally Nash*)

Consider the 2-player non-cooperative game where the two players want to maximize $J_1(x_1, x_2) := -\frac{1}{2}x_1^2 + x_1x_2$ w.r.t. x_1 and $J_2(x_1, x_2) := -\frac{1}{3}x_2^3 + x_1x_2$ w.r.t. x_2 , respectively.

- Show that the only potential Nash equilibria are $(0, 0)$ and $(1, 1)$.
- Show that both points are not Nash equilibria.
- Show, there is a neighborhood of $(1, 1)$ where this point qualifies as a Nash equilibrium.

EXERCISE 6.22 (*Symmetric game with infinite number of asymmetric Nash efficient actions*)

Consider the cost function $J(x_1, x_2) = x_1^2x_2^2$, where player i 's control instrument is x_i , $i = 1, 2$. Assume both players want to minimize $J(x_1, x_2)$.

- Show that this game has an infinite number of asymmetric Nash efficient actions.

- b. Show that this game has a unique set of symmetric Nash efficient actions.
- c. Show that this game has a unique Nash equilibrium.

EXERCISE 6.23 (Zero-sum game)

Consider a two-person game where player 1 wants to maximize $J(x_1, x_2)$ w.r.t. $x_1 \in \mathbb{R}^n$, and player two wants to minimize $J(x_1, x_2)$ w.r.t. $x_2 \in \mathbb{R}^m$, where $(x_1, x_2) \in \Gamma \subset \mathbb{R}^{n+m}$.

- a. Reformulate this problem into the zero-sum framework where both players want to maximize an objective function $J_i(x_1, x_2)$ w.r.t. x_i , $i = 1, 2$, respectively, where $J_1 + J_2 = 0$.
- b. Show that $(\hat{x}_1, \hat{x}_2)$ is Nash efficient for the zero-sum game with objective functions J_i , $i = 1, 2$, if and only if $(\hat{x}_1, \hat{x}_2)$ is a saddle-point equilibrium for J .
- c. Show that if Γ is a compact and convex set, and $J(x_1, x_2)$ is a continuous function which is moreover strictly concave in x_1 (for fixed feasible values of x_2) and strictly convex in x_2 (for fixed feasible values of x_1), the game has a Nash solution.
- d. Assume $x_i \in \Gamma_i$, where Γ_i , $i = 1, 2$, are convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Show, using item b., that if $(\hat{x}_1, \hat{x}_2)$ and $(\bar{x}_1, \bar{x}_2)$ are Nash efficient for above mentioned zero-sum game, also $(\hat{x}_1, \bar{x}_2)$ is Nash efficient for this game.
- e. Reconsider item c., with $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i , $i = 1, 2$, are compact and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Show that the zero-sum game has a unique Nash solution that is attained at a unique Nash efficient location. Hint: assume there are two different Nash efficient solutions, use next item d. to show that the value of J has to coincide at these points.
- f. Consider $\Gamma_i = [0, 1]$, $i = 1, 2$, and $J(x_1, x_2) = (x_1 - x_2)^2$. Show that the zero-sum game, where player 1 wants to maximize J w.r.t. x_1 and player 2 wants to minimize J w.r.t. x_2 has not a Nash equilibrium.

EXERCISE 6.24 (Symmetric zero-sum game)

Consider the objective function $J(x_1, x_2)$, with $x_i \in X \subset \mathbb{R}^m$. Assume $J(x_1, x_2) = -J(x_2, x_1)$. Assume player 1 wants to maximize $J(x_1, x_2)$ w.r.t. x_1 , while player 2 wants to minimize $J(x_1, x_2)$ w.r.t. x_2 .

- a. Show that this game is equivalent to a symmetric zero-sum game where player i wants to maximize $J_i(x_1, x_2)$ w.r.t. x_i , $i = 1, 2$, where $J_1(x_1, x_2) + J_2(x_1, x_2) = 0$ and $J_1(x_1, x_2) = J_2(x_2, x_1)$.
- b. Show that if this game has a set of Nash efficient actions $(\hat{x}_1, \hat{x}_2)$ also $(\hat{x}_2, \hat{x}_1)$ is a set of Nash efficient actions.
- c. Show that for any Nash efficient strategy $(\hat{x}_1, \hat{x}_2)$, $J_i(\hat{x}_1, \hat{x}_2) = 0$, $i = 1, 2$.
- d. Show that if this game has a set of Nash efficient actions $(\hat{x}_1, \hat{x}_2)$ also $(\hat{x}_1, \hat{x}_1)$ is a set of Nash efficient actions.

EXERCISE 6.25 (*Proof Theorem 6.6*)

Consider the notation in Theorem 6.6. Assume there exist $\hat{\alpha}_i \in (0, 1)$, with $\sum_{i=1}^N \hat{\alpha}_i = 1$, $\hat{\lambda}_i \geq 0$, $\hat{\mu}_i$ and $\hat{x} \in \Gamma$ such that $\hat{\lambda}_i(g_i(\hat{x}) - b_i) = 0$ and $L(x, \hat{\lambda}, \hat{\mu}) \leq L(\hat{x}, \hat{\lambda}, \hat{\mu}), \forall x \in \Gamma$. Use Theorem 6.4 to show that $\hat{x}$ is Pareto efficient.

EXERCISE 6.26 (*Proof Theorem 6.7*)

In this exercise you are asked to provide a proof for Theorem 6.7.

So, assume the set of constraint instruments Γ is convex; the objective function $J_i(x)$, player i wants to maximize, is concave, $i = 1, \dots, N$; and $\hat{x}$ is Pareto efficient.

- Show that there is no solution x satisfying simultaneously the set of inequalities $J_i(x) - J_i(\hat{x}) \geq 0$, $i = 1, \dots, N$, where at least one of the inequalities is strict.
- Use Lemma 5.8 to show there exist $\alpha \in \mathcal{A}$ such that $\sum_{i=1}^N \alpha_i J_i(x) \leq \sum_{i=1}^N \alpha_i J_i(\hat{x})$.

EXERCISE 6.27 (*Proof Corollary 6.13*)

Assume J_i , $i = 1, \dots, N$, are continuous differentiable on an open set $C \subset \mathbb{R}^n$. Let $x^* \in C$ be Pareto efficient for J_i , $i = 1, \dots, N$. Consider

$$L := \sum_{i=1}^N \lambda_i J_i(x).$$

- Use Lemma 6.10 and Theorem 4.35 to show there exist $\lambda_i \geq 0$, and $[\lambda_1, \dots, \lambda_N] \neq [0, \dots, 0]$ such that $\sum_{i=1}^N \lambda_i D_x J_i(x^*) = 0$.
- Show there exist $\alpha \in \mathcal{A}$ such that $\sum_{i=1}^N \alpha_i D_x J_i(x^*) = 0$.

EXERCISE 6.28 (*Bargaining solutions*)

In this exercise, we will consider three different bargaining solutions that are frequently discussed in the literature. You are asked to calculate all of them in the context of Example 6.8.

In bargaining theory, the central question is which outcome will be realized if the players decide to cooperate. In making this decision, the potential gains relative to the non-cooperative outcome play a key role. This non-cooperative outcome is typically referred to as the threat point or fallback position.

- Assume the players use the Nash solution as their fall-back position. Show that this solution is $d = (1, 1)$.
- In the Egalitarian solution players agree to divide the gains from cooperation equally. Geometrically, this is the intersection point of the 45 degrees line through the fall-back position with the Pareto frontier. Show that this solution is $(\frac{3}{2}, \frac{3}{2})$.
- In the solution proposed by Kalai and Smorodinsky first the ideal position is calculated. The best outcome player i can obtain when all other players play the non-cooperative solution is the i -th coordinate of the ideal position. Show that this ideal position is given by $I := (4\sqrt{\frac{1}{2}} - 1, 4\sqrt{\frac{1}{2}} - 1)$. Next they propose the intersection point on the Pareto frontier with the line connecting the ideal

and fall-back position as the resulting bargaining solution. Show this solution coincides with the Egalitarian solution.

- d. The third solution, proposed by Nash, is the solution on the Pareto frontier that maximizes the product of the gains from the fall-back position. That is the point (P_1, P_2) on the Pareto frontier for which $(P_1 - d_1)(P_2 - d_2)$ is maximal. Show this solution also coincides with the Egalitarian solution in this case.

EXERCISE 6.29 (Proof Theorem 6.26)

In this exercise you are asked to provide a proof for Theorem 6.26 in case there are 2 players.

So, assume the set of constraint instruments $\Gamma \subset \mathbb{R}^2$ is both compact and convex; and player i has instrument x_i at his disposal to maximize $J_i(x)$, $i = 1, 2$.

- Show that for every feasible instrument $(\bar{x}_1, \bar{x}_2) \in \Gamma$, the sets $\Gamma_1(\bar{x}_2) = \{(x_1, \bar{x}_2) \mid (x_1, \bar{x}_2) \in \Gamma\}$ and $\Gamma_2(\bar{x}_1) = \{(\bar{x}_1, x_2) \mid (\bar{x}_1, x_2) \in \Gamma\}$ are convex and compact.
- Show that if $J_1(x)$ is continuous on Γ , the reaction curve $\hat{x}_1(x_{-1})$ is well defined in the sense that at all $(\hat{x}_1(x_{-1}), x_{-1}) \in \Gamma$ the reaction curve has a value.
- Assume both reaction functions $(\hat{x}_i(x_{-i}))$ are continuous functions. Consider $f(x_1, x_2) := (\hat{x}_1(x_{-1}), \hat{x}_2(x_{-2}))$ on Γ . Use Brouwer's Fixed Point Theorem, see Exercise 2.57, to show that the game has a Nash equilibrium on Γ .

EXERCISE 6.30 (Proof Theorem 6.30)

In this exercise you are asked to provide a proof for Theorem 6.30.

So, assume the set of constraint instruments Γ is both compact and convex; the objective function $J_i(x)$, player i wants to maximize is continuous, and $J_i(x)$ is strictly concave w.r.t. its i^{th} instrument for every feasible instrument in Γ , $i = 1, \dots, N$.

- Show that for every feasible instrument $\bar{x} \in \Gamma$, the set Γ_i is convex and compact.
- Show that for every player i there exists a unique optimal response at every point $\bar{x}_i$ for which $\Gamma_i(\bar{x}_{-i})$ is not empty.
- Consider, for simplicity, the two-player scalar case. Show by contradiction that the reaction function $\hat{x}_1(x_2)$ is continuous. That is, assume the reaction function is not continuous at some point $\bar{x}_2$. This would imply that there exists a sequence x_2^N converging to $\bar{x}_2$ for which $\hat{x}_1(x_2^N)$ would not converge to $\hat{x}_1(\bar{x}_2)$. Use next the Theorem of Bolzano-Weierstrass which states that every sequence which values are bounded by some number b , has the property that this sequence has a subsequence s^n that converges, to arrive at a contradiction.
- Use Brouwer's Fixed Point Theorem, see Exercise 2.57, to show that there exists a Nash equilibrium.

EXERCISE 6.31 (Team problem)

Consider the problem where all N players share a common objective $J(x_1, \dots, x_N)$ and player i controls variable x_i , $i = 1, \dots, N$. We call $x^* = (x_1^*, \dots, x_N^*)$ team optimal if $J(x^*) \geq J(x)$, for all $x \in \Gamma$.

- Show that if x^* is team optimal, x^* is Nash efficient for the game where all players maximize $J(x)$.
- Consider the 2-player case with $J(x_1, x_2) = -x_1^2 - x_2^2 - 8x_1x_2 + 10x_1 + 10x_2$. Show that the corresponding game has a Nash equilibrium but not a team optimal solution.
- Show, using Lemma 6.10 and the definition of Pareto optimality, that x^* is team optimal if and only if x^* is Pareto efficient for the corresponding cooperative game.
- Consider the 2-player case with $J(x_1, x_2) = -(x_1 - 1)^2(x_2 - 2)^2$. Show that the corresponding team problem has an infinite number of team optimal locations.
- Consider the 2-player case with $J(x_1, x_2) = -(x_1 - x_2)^4 - \frac{4}{3}(x_1 - x_2)^3 + 4(x_1 - x_2)^2$. Show that the corresponding team problem has an infinite number of team optimal locations and has additional Nash efficient solutions that are not team optimal. Hint: consider the function $f(x) = -x^4 - \frac{4}{3}x^3 + 4x^2$.
- Consider a team problem that has an optimal solution. Motivate why the introduction of a leader into this problem will not improve its solution.

EXERCISE 6.32 Consider a game with a leader, L , and follower, F , who want to minimize their convex cost function $J_L(x_L, x_F)$ and $J_F(x_L, x_F)$ w.r.t. x_L and x_F , respectively. Assume J_L has a global minimum at $(x_L, x_F) = (a, b)$, J_F has a global minimum at $(x_L, x_F) = (c, d)$, and that both cost functions are differentiable. Assume, moreover, $D_1 J_F(a, b) \neq 0$.

- Show that the incentive strategy $x_L = \alpha x_F + \beta$, where $\alpha = -\frac{D_2 J_F(a, b)}{D_1 J_F(a, b)}$ and $\beta = a - \alpha b$ leads to a minimum cost for the leader.
- Use the result from item a) to verify the results of Exercise 6.12 and Example 6.38.

Problems with an Economic Content

EXERCISE 6.33

- Calculate the Nash equilibrium in case the cost function of firm i , $i = 1, 2$, is $C_1(x_1, x_2) = 8x_1 + 24$ and $C_2(x_1, x_2) = \frac{1}{2}x_2^2$, respectively. Assume the market price is given by $p = 92 - x_1 - x_2$.
- Assume welfare, W_i , in country i , $i = 1, 2$, can be measured as a function of the amount of pollution produced in both countries, x_i . Let $W_1 = -\frac{1}{3}(2x_1 + x_2 - 10)^3 + 2(2x_1 + x_2 - 10)^2 + 5(2x_1 + x_2 - 10)$ and $W_2 = -\frac{1}{2}(x_1 + 4x_2 - 12)^2 + 4(x_1 + 4x_2 - 12)$. Determine the Nash equilibrium of this game.
- Consider the next *pursuit-evasion game*. Person 1 is at position x and can run to position $x + u_1$, where $u_1 \in [0, \frac{1}{2}]$. Person 2 is at position y and can run to position $y + u_2$, where $u_2 \in [0, 1]$. Person 1 likes to catch person 2, whereas person 2 likes to avoid person 1. This problem is formalized as that person 1 likes to minimize the squared distance $(x + u_1 - (y + u_2))^2$ w.r.t. u_1 , whereas person 2 likes to maximize this squared distance w.r.t. u_2 . Calculate the Nash equilibrium of this game if the position of the persons is $(x, y) = (2, 1)$ and $(x, y) = (1, 2)$, respectively. Interpret both outcomes.

EXERCISE 6.34 (*competitive market entry by firms*)

Consider a market of N producers who produce the same good. Assume the price they get is $p = 92 - x$, where x is the total amount supplied by all producers together. The cost function of a producer is $C_i(x_i) = 32x_i + 3$.

- Calculate the Nash equilibrium if $N = 2$.
- Calculate the Nash equilibrium for an arbitrary N .
- How many producers will enter this market at most?

EXERCISE 6.35 (*subsidizing production green energy*)

Consider two electricity companies. The first company is a start-up company which intends to produce its electricity using "green" energy, the second company produces electricity using "fossil" energy. The price, p , they get on the market for electricity is: $p = 100 - x$, where x is the total amount of electricity supplied by both companies. The production cost for company 1 are initially $C(x_1) = 2x_1 + 650$ and for company two $C(x_2) = 4x_2 + 10$, respectively.

- Determine the profit function of both firms and show that, without a subsidy, firm one will not enter the market.
- Assume government is willing to pay firm one a loan to finance its debts until it reaches the break even point. Calculate how many years the government has to provide a loan. In how many years is company one able to pay the loan back to the government?
- Calculate the final price and electricity amounts by both companies that will occur in the market.
- Assume after a couple of years the fixed cost for the first company drop to 15. Additional "green" companies are allowed to enter the electricity market. Assume there are in total n "green" producing electricity companies in the market. Determine the equations the Nash solution has to satisfy. Show (using a computational package) that when 22 "green" producing electricity companies have entered the market the electricity company using "fossil" energy will drop out.

EXERCISE 6.36 Reconsider the fiscal policy coordination problem formulated in Exercise 6.4 item c.

- Calculate as well the Nash efficient equilibrium strategies, the corresponding output as the Nash equilibrium for an arbitrary $\bar{y}$. Show that for $\bar{y} = 1$ these numbers are approximately $(f_1^N, f_2^N) = (0.4948, 0.0493)$; $(y_1^N, y_2^N) = (4.4502, 1.1348)$ and $(J_1^N, J_2^N) = (0.5471, 0.0062)$, respectively.
- Show numerically that the Pareto solution in Exercise 6.4 item c. for which the distance of $(J_1(\alpha), J_2(\alpha))$ to $(0, 0)$ is as small as possible if $\bar{y} = 1$ is approximately $(J_1^m, J_2^m) = (0.5249, 0.0736)$ with $(f_1^m, f_2^m) = (0.4829, 0.2528)$ and $(y_1^m, y_2^m) = (4.4599, 1.2988)$.
- Show numerically that the Pareto solution in Exercise 6.4 item c. for which the distance to the Nash solution is as small as possible if $\bar{y} = 1$ is approximately $(J_1^{NB}, J_2^{NB}) = (0.5470, 0.0059)$ with $(f_1^{NB}, f_2^{NB}) = (0.5123, 0.0578)$ and $(y_1^{NB}, y_2^{NB}) = (4.4669, 1.1487)$.
- Comment on the possibility that fiscal authorities will coordinate their policies in order to achieve their goals.

EXERCISE 6.37 (*tracking an individual variable within a large population*)

Consider the problem where every individual i likes to track a variable y_i towards some target value $\bar{y}_i$ within a large population. The impact individuals have on each other's variable is assumed to be negligible. But the aggregate impact of the population to every player's variable is supposed to be significant. Below you are asked to analyze the noncooperative Nash outcome of this game under the assumption that the cost of using a control effort of x_i by player i to manipulate the variable y_i is growing quadratically.

We model this problem as follows:

Player i wants to minimize $J_i(x_1, \dots, x_N) := (y_i - \bar{y}_i)^2 + r_i x_i^2$, where $y_i = ax_i + \frac{1}{N} \sum_{j=1}^N x_j + d_i$, $i = 1, \dots, N$.

Here $r_i > 0$ is some constant which indicates the preference of player i of reaching his target versus the control efforts he has to use. The constant d_i can be interpreted as the current value of the variable y_i which player i likes to control towards $\bar{y}_i$ using a control effort x_i , and $a + \frac{1}{N}$ models the direct impact of control on the target variable.

a. Show that x_i , $i = 1, \dots, N$, are Nash efficient for this game if and only if they satisfy next equations:

$$N(a + \frac{1}{N} + \frac{r_i}{a + \frac{1}{N}})x_i + \sum_{j \neq i} x_j = N(\bar{y}_i - d_i), \quad i = 1, \dots, N.$$

b. Assume throughout $r_i = r$ for all i . Let $s := N(a + \frac{1}{N} + \frac{r}{a + \frac{1}{N}})$. Show that

$$x_i = \frac{N}{s(s + N - 2) - N + 1}((s + N - 1)(\bar{y}_i - d_i) - \sum_{j=1}^N (\bar{y}_j - d_j)), \quad i = 1, \dots, N,$$

solve the equations advertized in item a.

c. Show that for a large population, the action of player i converges to:

$$x_i^c = \frac{a}{(a^2 + r)(a^2 + a + r)}((a^2 + a + r)(\bar{y}_i - d_i) - ae_{avg}), \quad \text{where } e_{avg} = \frac{1}{N} \sum_{i=1}^N (\bar{y}_i - d_i)$$

and variable y_i converges to

$$y_i^c = \frac{a^2}{a^2 + r} \bar{y}_i + \frac{r}{a^2 + r} d_i + \frac{1}{N} \sum_{j=1}^N \frac{a}{a^2 + r} (\bar{y}_j - d_j) - e_{avg} \frac{a^3 + a^2}{(a^2 + r)(a^2 + a + r)}.$$

d. Show that if either players do not care about the amount of used control or the direct impact of control on the target variable is high (i.e., either r is almost zero or a is large), the target variable is (almost) realized and the control used by the players converges to $x_i = \frac{1}{a}(\bar{y}_i - d_i) - \frac{1}{a(a+1)}e_{avg}$. This, under the assumption that the target values and current values of the variables are not very high.

- e. Assume the current value (initial position) of all players is the same, $d_i = d$, and half of the (large) population is reaching for a high target value of y^h and the other half of the population likes to target the variable y_i at a low value of y^l .

Calculate the approximate value of the instruments both sets of players use and show that the difference between the approximate output variables reaching for a high target and low target is $\frac{a}{a^2+r}(y^h - y^l)$.

Discuss the impact of control cost on above result and the direct impact of control on the realization of the target variable.

EXERCISE 6.38 (*aligning the population average with individual preferences in a large population*)

Consider the problem where every individual i likes to track a variable y towards some target value $\bar{y}_i$ within a large population. The impact individuals have on the target variable is assumed to be negligible. But the aggregate impact of the population is supposed to be significant. Below you are asked to analyze the noncooperative Nash outcome of this game under the assumption that the cost of using a control effort of x_i by player i to manipulate the variable y is growing quadratically.

We model this problem as follows:

Player i wants to minimize $J_i(x_1, \dots, x_N) := (y - \bar{y}_i)^2 + r_i x_i^2$, where $y = \frac{1}{N} \sum_{i=1}^N x_i + d$, $i = 1, \dots, N$. Here $r_i > 0$ is some constant which indicates the preference of player i of reaching his target versus the control efforts he has to use. The constant d can be interpreted as the current value of the variable y which player i likes to control towards $\bar{y}_i$ using a control effort x_i .

- a. Show that x_i , $i = 1, \dots, N$, are Nash efficient for this game if and only if they satisfy next equations:

$$(1 + N^2 r_i) x_i + \sum_{j \neq i} x_j = N(\bar{y}_i - d), \quad i = 1, \dots, N.$$

- b. Show that for every individual player the gap between the target and realization of the target variable, $\bar{y}_i - y$, equals $N r_i x_i$.
- c. Next assume all players have the same aversion to using control, i.e., $r_i = r$, $\forall i$. Let $s := 1 + N^2 r$. Show that

$$x_i = \frac{N}{s(s + N - 2) - N + 1} ((s + N - 1)(\bar{y}_i - d) - \sum_{j=1}^N (\bar{y}_j - d)), \quad i = 1, \dots, N,$$

solve the equations advertized in item a.

- d. Use items b. and c. to show that $\bar{y}_i - y = \bar{y}_i - d - \frac{1}{N(Nr+1)} \sum_{j=1}^N (\bar{y}_j - d)$. Assume $r \neq 0$ and the average value $\frac{1}{N} \sum_{j=1}^N (\bar{y}_j - d)$ converges. Show that the larger the population becomes, the target variable is almost not influenced by the control anymore. Or, stated differently, for large populations no tracking occurs.
- e. Assume half of the population is reaching for a high target value of y^h , with $r_i = r_h$, and the other half of the population likes to target the variable y_i at a low value of y^l , with $r_i = r_l$. Moreover,

assume both groups combine "forces" to reach their target value. That is, within both groups all control instruments are used collectively (but symmetrically) to reach individual's targets.

Show that at the Nash solution of this "collective" game $y = \frac{1}{4r_h r_l + r_h + r_l} (r_l(\bar{y}_h - d) + r_h(\bar{y}_l - d)) + d$.

- f. Show that in item e., the fraction $\frac{r_h}{r_l}$ determines to what extent a population group will succeed in the realization of its goal.

EXERCISE 6.39 (taxing import industry by government)

Consider two countries. In both countries there is a firm exporting goods to the other country. Assume firm 2 produces in country 2 flowers that it exports to country 1, whereas firm 1 produces oil in country 1 which it exports to country 2. Assume the demand for flowers in country 1, q_1 , is given by $q_1 = \bar{q}_1 - \alpha_1(p_1 + t_1) + \beta\bar{y}_1$ and the demand for oil in country 2, q_2 by $q_2 = \bar{q}_2 - \alpha_2(p_2 + t_2) + \beta\bar{y}_2$. Here, $\bar{y}_i$ denotes national income and t_i the tax raised by country i , $i = 1, 2$. Furthermore, p_1 is the price of flowers determined in country 2 by firm 2 and p_2 is the price of oil determined by firm 1 in country 1. The $\bar{q}_i$ may be interpreted as representing exogenous demand for the product i in both countries.

Assume firm 1 wants to maximize its profit $\pi_1 := p_2 q_2 - \gamma_1 q_2$. So, firm 1's cost of oil production are assumed to be linearly dependent on its demand. Firm 2 wants to maximize its profit $\pi_2 := p_1 q_1 - \gamma_2(p_2 + t_2)q_1$. The production of flowers is taking place in green houses that are heated using oil. Therefore its cost are linearly related to the oil price times the quantity sold.

Finally it is assumed that governments in both countries want to maximize on the one hand tax income and on the other hand sustain their exporting industry. Which is formalized as they determine tax that maximize a weighted average of both objectives, i.e. to maximize $\rho_i \pi_i + t_i q_i$, $i = 1, 2$, respectively.

- a. Show that the Nash equilibrium of this game is given by:

$$\begin{bmatrix} p_1 \\ t_1 \\ p_2 \\ t_2 \end{bmatrix} = \frac{1}{d} \begin{bmatrix} \alpha_2(\alpha_1 s_1 \rho_2 \gamma_2^2 - 3\alpha_2 s_1 - 4\alpha_1 s_2 \gamma_2 - 2\alpha_1 \alpha_2 \gamma_1 \gamma_2) \\ \alpha_2(2\alpha_1 s_2 \gamma_2 + \alpha_1 \alpha_2 \gamma_1 \gamma_2 - 3\alpha_2 s_1) \\ \alpha_1(\alpha_1 \alpha_2 \rho_2 \gamma_1 \gamma_2^2 + \alpha_1 s_2 \rho_2 \gamma_2^2 - 3\alpha_2 s_2 - \alpha_2 s_1 \rho_2 \gamma_2 - 6\alpha_2^2 \gamma_1) \\ \alpha_1(3\alpha_2^2 \gamma_1 - \alpha_1 \alpha_2 \rho_2 \gamma_1 \gamma_2^2 + 2\alpha_2 s_1 \rho_2 \gamma_2 - 3\alpha_2 s_2 - \alpha_1 s_2 \rho_2 \gamma_2^2) \end{bmatrix},$$

where $s_i := \bar{q}_i + \beta_i \bar{y}_i$, $i = 1, 2$, and $d = \alpha_1 \alpha_2 (\alpha_1 \rho_2 \gamma_2^2 - 9\alpha_2)$.

- b. Assume government 1 is the leader in the game and the other 3 players play Nash.

Show that in principle government 1 has the incentive to raise taxes ad infinitum. Show that the best actions where the economy still survives are given by:

$$\begin{bmatrix} p_1 \\ p_2 \\ t_2 \end{bmatrix} = \frac{1}{3\alpha_1 \alpha_2} \begin{bmatrix} \alpha_2(3s_1 - 6\alpha_1 t_1) \\ \alpha_1(2\alpha_2 \gamma_1 + s_2 + \alpha_1 \rho_2 \gamma_2 t_1) \\ \alpha_1(s_2 - \alpha_2 \gamma_1 - 2\alpha_1 \rho_2 \gamma_2 t_1) \end{bmatrix},$$

where $t_1 = \frac{2\gamma_2 s_2 + \alpha_2 \gamma_1 \gamma_2 - 3\alpha_2 s_1}{\alpha_1 \rho_2 \gamma_2^2 - 6\alpha_1 \alpha_2}$ (assuming parameters are such that $t_1 > 0$).

c. Next assume Government 2 announces that it will levy the same taxes as country 1.

Calculate the Nash efficient solutions for the resulting 3-player game.

d. Assume that, under the conditions of item c., Government 1 is the leader in the 3-player game.

Show that a unique efficient solution exists, assuming the two firms play Nash if $\alpha_1(1 + \frac{\gamma_2}{2}) > \frac{\alpha_2 \rho_1}{2}$.

Under which additional conditions will this solution be realized?

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This book reviews a range of standard mathematical tools used to solve optimization problems.

It discusses the conditions under which function optima exist, considering both single-variable and multi-variable functions. Because resources are often limited, special attention is given to constrained optimization problems, where the domain of the function is restricted. Convex and linear programming problems are examined in greater detail due to their favorable numerical properties.

Finally, the book addresses a number of concepts related to optimization problems involving multiple agents or multiple objectives.

The main text is accessible to students of economics as well as to those pursuing more technically oriented fields of study.

Its aim is to provide students with a clear understanding of the underlying concepts, an intuitive grasp of the conditions under which specific mathematical tools can be applied, and practical guidance on their use. Mathematical details required for full theoretical rigor are largely omitted from the main text, but additional rigor is provided through supplementary exercises.

Jacob Engwerda was born in Nieuweschans, the Netherlands, in 1958. He obtained his master's degree from the University of Groningen in 1982 and earned his Ph.D. in mathematics, with a specialization in systems and control theory, from the Eindhoven University of Technology in 1988.

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